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Journal of Geometry and Physics 51 (2004) 40–70

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Generalized BRST models and topological Yang–Mills theories

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Received 31 May 2003

Abstract

We present a general framework for models admitting a decomposition of the type $d = [\delta, b]$, with b the BRST operator and δ a certain (even) derivation. We focus our attention on models whose fields can be described as components of two ladders $\mathcal{W} = c + A + \cdots$ and $\mathcal{F} = \phi + \psi + \cdots$ and show how they relate to some aspects of topological Yang–Mills theory. We relate our construction to the standard mathematical ideas of Cartan's $\mathcal{G}$-operation and interpret $\mathcal{W}$ and $\mathcal{F}$ as pair of algebraic connection and curvature in a certain bigraded differential algebra.

MSC: 81T13; 81T45

JGP SC: Quantum field theories

Keywords: Topological field theory; BRST algebras

1. Introduction

In this work we intend to investigate a class of models defined by ladders of the type:

$$\mathcal{W} \equiv \sum_{i=0}^{D} \varphi_i^{1-i} = c + A + \sum_{i=2}^{D} \varphi_i^{1-i}, \tag{1}$$

$$\mathcal{F} \equiv \sum_{i=0}^{D} \eta_i^{2-i} = \phi + \psi + B + \sum_{i=3}^{D} \eta_i^{2-i} \tag{2}$$

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doi:10.1016/j.geomphys.2003.09.003

containing the basic fields $\{c, A, \phi, \psi\}$ of TYMT as given in [1–3]. These ladders satisfy connection-curvature like equations:

$$\tilde{d}\mathcal{W} + \tfrac{1}{2}[\mathcal{W}, \mathcal{W}] = \mathcal{F}, \tag{3}$$

$$\tilde{d}\mathcal{F} + [\mathcal{W}, \mathcal{F}] = 0 \tag{4}$$

with

$$\tilde{d} = b + d + \sum_{i=2}^{D} \Delta_i^{1-i}. \tag{5}$$

In this formulation, the presence of high component fields φ_i^{1-i}, η_i^{2-i} in the ladders $\mathcal{W}$, $\mathcal{F}$, and of additional operators Δ_i^{1-i} in the general derivative $\tilde{d}$ offers an attempt to extend the superfield approach of TYMT originally introduced in [2]. Here, an object written as X_i^j is supposed to have bidegree (i, j) where i denotes form degree and j the ghost number. The operators Δ_i^{1-i} are superderivations that acting on a field X_k^r produce a field with bidegree $(i+k, r+1-i)$. The field B is a two-form, generally not depending on the curvature of A, $F = dA + A^2$. The general derivative $\tilde{d}$ contains the BRST operator b, which is determined from (3) and (4) after expanding these equations in terms with same form degree. The operator d denotes the exterior derivative.

One motivation for the study of such models is to look for possible extensions of the Chern–Simons term, the gauge anomaly and the Donaldson polynomials. The extensions of the Chern–Simons term and the gauge anomaly were developed in [4] for a model defined by D-dimensional ladders $\mathcal{W} = c + A + \varphi_2^{-1} + \cdots + \varphi_D^{1-D}$, $\mathcal{F} = \phi + \psi + B + \eta_3^{-1} + \cdots + \eta_D^{2-D}$ and derivative $\tilde{d} = b + d$. The power of this formulation is that it allows to encode in a single model both expressions for the Chern–Simons term and the gauge anomaly.

As for the Donaldson polynomials, the strategy is to consider descent equations:

$$\begin{aligned} &b\omega_4^0 + d\omega_3^1 = 0, \qquad b\omega_3^1 + d\omega_2^2 = 0, \qquad b\omega_2^2 + d\omega_1^3 = 0,\\ &b\omega_1^3 + d\omega_0^4 = 0, \qquad b\omega_0^4 = 0 \end{aligned} \tag{6}$$

with the cycles ω_i^{4-i} $(0 \le i \le 4)$ being polynomials in the functional space $\mathcal{V} = \{c, A, \varphi_i^{1-i}, \phi, \psi, B, \eta_i^{2-i}; dc, dA, d\varphi_i^{1-i}, d\phi, d\psi, dB, d\eta_i^{2-i}\}$. When we consider a simple model, defined on the functional space $\mathcal{V} = \{c, A, \phi, \psi, dc, dA, d\phi, d\psi\}$, one finds the generators of Donaldson polynomials [1–5] as a possible solution to the descent equations, i.e.:

$$\begin{aligned} &\omega_0^4 = \mathrm{Tr}(\tfrac{1}{2}\phi^2), \qquad \omega_1^3 = \mathrm{Tr}(\phi\psi), \qquad \omega_2^2 = \mathrm{Tr}(\phi F + \tfrac{1}{2}\psi^2),\\ &\omega_3^1 = \mathrm{Tr}(\psi F), \qquad \omega_4^0 = \mathrm{Tr}(\tfrac{1}{2}F^2). \end{aligned} \tag{7}$$

As it was shown in [5], for a model with ladders $\mathcal{W} = c + A$, $\mathcal{F} = \phi + \psi$ and differential $\tilde{d} = b + d + \Delta_2^{-1} + \Delta_3^{-2} + \Delta_4^{-3}$ we have obtained solutions $\omega_i^{4-i} \equiv \omega_i^{4-i}(\alpha_1, \ldots, \alpha_8)$, which reduce to (7) when the parameters $(\alpha_1, \ldots, \alpha_8)$ are set to zero. The interesting aspect of this solution is that it shows the existence of other quantum field theory models providing a description for the Donaldson polynomials that differs from the approach of [1–3].

The purpose of our study is twofold. First, we intend to complete the study of models described by ladders (1) and (2) [4–7] by considering the case of negative ghost number fields and a general derivative as in (5). Thus, we expect that the presence of negative ghost number fields, the field B and operators Δ_i^{1-i} will modify the solution (7) giving a generalization for the Donaldson polynomials for a model described by (1), (2) and (5). In general, even though these extensions may not define interesting topological invariants, they still contain the terms associated to the generators of Donaldson polynomials (see Eqs. (110)–(114)).

Second, we try to put our work into a general perspective by showing how an appropriate choice of ladders and derivative $\tilde{d}$ allow us to describe several distinct models, e.g. Yang–Mills, TYMT, Chern–Simons, BF, etc. In this respect, our model is a particular case of a *superfield* formalism which consists on accommodating gauge fields, ghosts, antighosts, etc. as component of certain ladders. Essentially, these models can be divided into two categories: (I) those admitting ladders satisfying connection-curvature like equations (e.g. [2–10]); and (II) those where this requirement is absent (e.g. [11–14]). The ideas underlying the models in category (I) constitute a general approach for determining the BRST transformations for a set of fields given that Eqs. (3)–(5) are satisfied for a certain choice of ladders $\mathcal{W}$, $\mathcal{F}$ and derivative $\tilde{d}$. In these models, the general derivative contains *at least* the BRST operator and the exterior derivative, while the ladders may contain several others component fields. The combined use of extended ladders and derivatives has found applications in many different models (see, for example, the recent development of [10] for the stochastic quantization of Yang–Mills theory in five dimensions, and [5,7] for the description of TYMT and four-dimensional Yang–Mills theory).

The main feature of our model lies on the existence of a $(1,-1)$ derivation δ that allows us to exhibit a *particular* solution for the descent equations (6) once we have solved $b\omega_0^4 = 0$. Mathematically, δ converts a problem of determining the cohomology of b modulo d into a simple one, the cohomology of b alone. It was in this context that δ has originally appeared in [15], and since then it has been successfully applied in the algebraic renormalization of several models [16,17]. Formally, we define δ through Eqs. (26)–(28). In particular, from (28) we obtain the form of the operators Δ_i^{1-i} as given in (31), and condition $d = [\delta, b]$. The δ operator is closely related to the so-called VSUSY symmetry discovered in the quantization of Chern–Simons [18,19] and BF topological theories [20]. This symmetry is determined by an odd derivation δ_τ parameterized by a vector field $\tau = \tau^\mu \partial_\mu$, and it satisfies an equation of the type[1] $[\delta_\tau, b] = \mathcal{L}_\tau$ [21] with $\mathcal{L}_\tau$ the Lie derivative along τ. Another common aspect is that many VSUSY models are formulated adopting a superfield formalism [19–22], which resembles (3)–(5). Nonetheless, in all these models the VSUSY operator δ_τ is not restricted by (26)–(28).

From a mathematical point of view, it is difficult to adopt the interpretation of [2,3] and consider the negative ghost number fields as components of a curvature and connection on the G-bundle[2] $((P \times \mathcal{C})/\underline{G}, M \times \mathcal{C}/\underline{G})$. In addition, the operators Δ_i^{1-i} cannot be interpreted as components of a general derivative in this bundle. This lead us to look for another description.

[1] In the literature of VSUSY there are some modifications on the form assumed by $[\delta_\tau, b]$.

[2] $\mathcal{C}$ and $\underline{G}$ denotes, respectively, the space of connections and the group of gauge transformations on a principal fiber bundle P.

One possibility is to use the construction of BRST differential algebras as given by Dubois-Violette [8,9]. This treatment has been applied successfully in [5] for a model containing only positive ghost number fields and the operators Δ_i^{1-i}. Our task here is to introduce in a consistent way negative ghost number fields into the approach of BRST differential algebras used in [5,8,9]. We recall that, even before the formulation of TQFT, the two lowest components c, A of $\mathcal{W}$ were already geometrically understood as the Maurer–Cartan form on the group of gauge transformations [23] and a connection one-form on a principal bundle. Therefore, since c is a field with ghost number one, it will be considered here as a one-form on the group of gauge transformations. We cannot think of φ_i^{1-i} ($i \geq 2$) as a $(i-1)$-form on the same space. In fact, if φ_i^{1-i} were a $(i-1)$-form on the same space as c it would be natural to take the multiplication between them as the exterior product of forms. Then, $c \wedge \varphi_i^{1-i}$ would be a i-form. Nonetheless, the additive Z-graded structure (associated to the ghost number) of the space which they belong would force $c \wedge \varphi_i^{1-i}$ to be a $(i-2)$-form. Therefore, we will have an ambiguity if we consider the positive and negative ghost number fields belonging to the same space. The solution is to define the negative ghost number field φ_i^{1-i} as a $(i-1)$-form on the dual of the algebra of the group of gauge transformations. A similar argument shows that the negative ghost number fields η_i^{2-i} should be defined as $(i-2)$-forms on this same space.

The other problem, on the meaning of $\tilde{d}$, is solved as a consequence of the first one, e.g. once we know the space $\mathcal{K}^{(m,n)}$ (m and n labeling, respectively, form degree and ghost number) each of the fields in $\mathcal{W}$ and $\mathcal{F}$ belongs, we can define a space $\mathcal{K} = \oplus_{(m,n)}\mathcal{K}^{(m,n)}$ on which $\tilde{d}$ acts as a derivation. Indeed, we will see that $\mathcal{K} = \oplus_{(m,n)\in Z^+\times Z}\mathcal{K}^{(m,n)}$ will have the structure of a bigraded differential algebra with $\mathcal{K}^{(m,n)}$ being the space of n-linear antisymmetric maps on $\underline{\mathcal{G}}$ or $\underline{\mathcal{G}}^*$, polynomial in $\mathcal{C}$ and with values in $\Omega^m(P)$, i.e. $\mathcal{K}^{(m,n)} = \mathcal{F}(\mathcal{C}\times\underline{\mathcal{G}}^n, \Omega^m(P)) \simeq \mathcal{F}(\mathcal{C}, \bigwedge^n \underline{\mathcal{G}}^* \otimes \Omega^m(P))$ if $n > 0$ and $\mathcal{K}^{(m,n)} = \mathcal{F}(\mathcal{C}\times\underline{\mathcal{G}}^{*n}, \Omega^m(P)) \simeq \mathcal{F}(\mathcal{C}, \bigwedge^n \underline{\mathcal{G}} \otimes \Omega^m(P))$ if $n < 0$. Here, $\underline{\mathcal{G}}$ denotes the Lie algebra of the group of gauge transformations, $\Omega(P)$ the space of forms in P and $\mathcal{C}$ the space of connections on P. The ladders $\mathcal{W}$ and $\mathcal{F}$ will be elements of a subalgebra $\mathcal{H} \subset \mathcal{K}$ that is generated by the fields $\varphi_i^{1-i}, d\varphi_i^{1-i}, \eta_i^{2-i}, d\eta_i^{2-i}, i \geq 0$.

Our work is organized as follows. In Section 2 we introduce two generalized ladders $\mathcal{W}$, $\mathcal{F}$ whose components will accommodate the fields of our model. We impose the ladders satisfy a couple of connection-curvature like equations that will be related to the BRST transformations of the fields. We adopt a step-by-step procedure for determining the BRST transformations, we introduce the δ operator, determine Δ_i^{1-i} and all constraints they satisfy. In Section 3 we discuss a four-dimensional model with ladders of the type $\mathcal{W} = c + A + \varphi_2^{-1}$, $\mathcal{F} = \phi + \psi + B$ and differential $\tilde{d} = b + d + \Delta_2^{-1} + \Delta_3^{-2} + \Delta_4^{-3}$. We analyze how the expression for the Donaldson polynomials are modified by the presence of the fields φ_2^{-1}, B and the operators $\Delta_2^{-1}, \Delta_3^{-2}, \Delta_4^{-3}$. In Section 4 we show how the original zero-curvature models of [6,7] are obtained as a particular case of imposing $\mathcal{F} = 0$. In Section 5 we give a mathematical interpretation of our model. We relate our construction to the set up of BRST algebras following closely the approach developed in [8,9]. We review the concepts of gauge group and gauge algebra, and finally present an explicit realization of our model in terms of the algebra of differential forms on a principal fiber bundle P.

2. Constructing the model

Let G be a Lie group and $\mathcal{G}$ its Lie algebra whose generators we denote by $\{e_a\}$ ($a = 1, \dots, \dim G$). We denote the product $e_{a_1} \cdots e_{a_n} := \gamma^c_{a_1 \cdots a_n} e_c$ with $\gamma^c_{a_1 \cdots a_n} \in K$ ($K = R$ or C). Let us consider a set of fields and its derivatives $\{\varphi_i^{1-i}, d\varphi_i^{1-i}, \eta_j^{2-j}, d\eta_j^{2-j}\}$, $0 \leq i, j \leq D$ with the upper and lower indices labeling, respectively, ghost number and form degree. At this point, those fields are considered as Lie algebra valued maps defined on a generic spacetime $\mathcal{M}$. We denote by $\mathcal{V}$ the space of local polynomials in the fields and their derivatives. The total degree of a field is given by the sum of its form degree and ghost number. We say that $\alpha \in \mathcal{V}$ is a homogeneous element of bidegree (m, n) if it is written as a sum of terms with form degree m and ghost number n. The total degree of a homogeneous element of type (m, n) is then $m+n$. Given two homogeneous elements of bidegrees (m, n), (p, q), $\alpha^n_m, \beta^q_p \in \mathcal{V}$, we define the Lie-bracket: $[\alpha, \beta] \doteq \alpha\beta - (-1)^{(m+n)(p+q)}\beta\alpha$.

2.1. The BRST transformations

Let $\mathcal{W}$, $\mathcal{F}$ and $\tilde{d}$ be given by (1), (2) and (5) and satisfying (3) and (4). Expanding (3) and (4) in terms with same form degree we obtain (we adopt the convention $\Delta^1_0 := b$, $\Delta^0_1 := d$):

$$b\varphi_k^{1-k} + d\varphi_{k-1}^{2-k} + \sum_{i=2}^{k} \Delta_i^{1-i}\varphi_{k-i}^{1-k+i} + \frac{1}{2}\sum_{i=0}^{k}[\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}] - \eta_k^{2-k} = 0, \quad 0 \leq k \leq D, \tag{8}$$

$$b\eta_k^{2-k} + d\eta_{k-1}^{3-k} + \sum_{i=2}^{k} \Delta_i^{1-i}\eta_{k-i}^{2-k+i} + \sum_{i=0}^{k}[\varphi_i^{1-i}, \eta_{k-i}^{2-k+i}] = 0, \quad 0 \leq k \leq D, \tag{9}$$

$$\sum_{i=0}^{k} \Delta_i^{1-i}\Delta_{k-i}^{1-k+i} = 0, \quad 0 \leq k \leq D. \tag{10}$$

Let us now suppose that it exists $q, p \in N$, $2 \leq q \leq D$, $2 \leq q \leq D$ (the case $q = 1$ was studied in [5]) such that

$$\varphi_i^{1-i} = \begin{cases} 0 & \text{if } i > q, \\ \neq 0 & \text{if } i \leq q \end{cases} \quad \text{and} \quad \eta_j^{2-j} = \begin{cases} 0 & \text{if } j > p, \\ \neq 0 & \text{if } j \leq p. \end{cases}$$

Then (8) and (9) break into:

$$b\varphi_k^{1-k} = -d\varphi_{k-1}^{2-k} - \sum_{i=2}^{k} \Delta_i^{1-i}\varphi_{k-i}^{1-k+i} - \frac{1}{2}\sum_{i=0}^{k}[\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}] + \eta_k^{2-k}, \quad 0 \leq k \leq q, \tag{11}$$

$$\sum_{i=2}^{q+1} \Delta_i^{1-i}\varphi_{q+1-i}^{-q+i} = -d\varphi_q^{1-q} - \frac{1}{2}\sum_{i=1}^{q}[\varphi_i^{1-i}, \varphi_{q+1-i}^{-q+i}] + \eta_{q+1}^{1-q}, \tag{12}$$

$$\sum_{i=k-q}^{k} \Delta_i^{1-i} \varphi_{k-i}^{1-k+i} = -\frac{1}{2} \sum_{i=k-q}^{q} [\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}] + \eta_k^{2-k}, \quad k \geq q+2, \tag{13}$$

$$b\eta_k^{2-k} = -d\eta_{k-1}^{3-k} - \sum_{i=2}^{k} \Delta_i^{1-i} \eta_{k-i}^{2-k+i} - \sum_{i=0}^{k} [\varphi_i^{1-i}, \eta_{k-i}^{2-k+i}], \quad 0 \leq k \leq p, \tag{14}$$

$$\sum_{i=2}^{p+1} \Delta_i^{1-i} \eta_{p+1-i}^{-p+1+i} = -d\eta_p^{2-p} - \sum_{i=1}^{p+1} [\varphi_i^{1-i}, \eta_{p+1-i}^{-p+1+i}], \tag{15}$$

$$\sum_{i=2}^{k} \Delta_i^{1-i} \eta_{k-i}^{2-k+i} = -\sum_{i=0}^{k} [\varphi_i^{1-i}, \eta_{k-i}^{2-k+i}], \quad k \geq p+2. \tag{16}$$

Eqs. (11) and (14) cannot be taken as the BRST transformations of the fields unless we specify the form of the operators Δ_i^{1-i} ($i \geq 2$) on their right-hand side. One way of dealing with this is to impose

$$\sum_{i=2}^{k} \Delta_i^{1-i} \varphi_{k-i}^{1-k+i} = 0, \quad 0 \leq k \leq q, \tag{17}$$

$$\sum_{i=2}^{k} \Delta_i^{1-i} \eta_{k-i}^{2-k+i} = 0, \quad 0 \leq k \leq p, \tag{18}$$

which then fix the BRST transformations as

$$b\varphi_k^{1-k} = -d\varphi_{k-1}^{2-k} - \frac{1}{2} \sum_{i=0}^{k} [\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}] + \eta_k^{2-k}, \quad 0 \leq k \leq q, \tag{19}$$

$$b\eta_k^{2-k} = -d\eta_{k-1}^{3-k} - \sum_{i=0}^{k} [\varphi_i^{1-i}, \eta_{k-i}^{2-k+i}], \quad 0 \leq k \leq p. \tag{20}$$

2.2. *Implementing the conditions* $[b, d] = 0$ *and* $b^2 = 0$

Let us now consider (10). Taking $k = 0$ and $k = 1$ we obtain $b^2 = 0$ and $[b, d] = 0$. These two conditions should be satisfied on the set $\{\varphi_i^{1-i}, d\varphi_i^{1-i}, \eta_j^{2-j}, d\eta_j^{2-j}\}$ ($0 \leq i \leq q, 0 \leq j \leq p$). Implementing the condition $[b, d] = 0$ fixes the BRST transformation of the field derivatives:

$$bd\varphi_k^{1-k} = \sum_{i=0}^{k} [d\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}] - d\eta_k^{2-k}, \quad 0 \leq k \leq q, \tag{21}$$

$$bd\eta_k^{2-k} = \sum_{i=0}^{k} [d\varphi_i^{1-i}, \eta_{k-i}^{2-k+i}] - \sum_{i=0}^{k} [\varphi_i^{1-i}, d\eta_{k-i}^{2-k+i}], \quad 0 \leq k \leq p. \tag{22}$$

The nilpotency of b is satisfied on the fields φ_i^{1-i}, $0 \leq i \leq q$ with no further restriction, but over η_i^{2-i}, $0 \leq i \leq p$ we obtain

$$\begin{aligned} b^2\eta_k^{2-k} &= \frac{1}{2}\sum_{i=0}^{\epsilon(q,k)}\sum_{r=0}^{i}[[\varphi_r^{1-r},\varphi_{i-r}^{1-i+r}],\eta_{k-i}^{2-k+i}] - \sum_{i=0}^{\epsilon(q,k)}\sum_{r=0}^{i}[\varphi_{k-i}^{1-k+i},[\varphi_r^{1-r},\eta_{i-r}^{2-i+r}]] \\ &\quad - \sum_{i=0}^{\epsilon(q,k)}[\eta_i^{2-i},\eta_{k-i}^{2-k+i}], \quad 0 \leq k \leq p \end{aligned} \tag{23}$$

with $\epsilon(q,k) \equiv \min\{q,k\}$ being the minimum element between q and k. Since the fields are independent, in order to obtain $b^2\eta_k^{2-k} = 0$ we should have

$$0 = \frac{1}{2}\sum_{i=0}^{\epsilon(q,k)}\sum_{r=0}^{i}[[\varphi_r^{1-r},\varphi_{i-r}^{1-i+r}],\eta_{k-i}^{2-k+i}] - \sum_{i=0}^{\epsilon(q,k)}\sum_{r=0}^{i}[\varphi_{k-i}^{1-k+i},[\varphi_r^{1-r},\eta_{i-r}^{2-i+r}]], \tag{24}$$

$$0 = \sum_{i=0}^{\epsilon(q,k)}[\eta_i^{2-i},\eta_{k-i}^{2-k+i}]. \tag{25}$$

The only way to vanish (25) without imposing any constraint on the fields η_i^{2-i} is to take $\epsilon(q,k) = k$. With this choice, and using Jacobi identity, we have also satisfied (24).

Since the condition $\epsilon(q,k) = k$ must be verified for all values of k within $0 \leq k \leq p$ we obtain the constraint $p \leq q$.

2.3. *Determination of* Δ_i^{1-i}, $i \geq 2$

The operators Δ_i^{1-i} are determined through the introduction of an operator δ of bidegree $(1,-1)$ such that

$$\mathcal{W} = \mathrm{e}^{\delta}c, \tag{26}$$

$$\mathcal{F} = \mathrm{e}^{\delta}\phi, \tag{27}$$

$$\tilde{d} = \mathrm{e}^{\delta}b\,\mathrm{e}^{-\delta}. \tag{28}$$

These equations are equivalent to

$$\delta\varphi_k^{1-k} = (k+1)\varphi_{k+1}^{-k}, \quad 0 \leq k \leq q, \tag{29}$$

$$\delta\eta_k^{2-k} = (k+1)\eta_{k+1}^{-k+1}, \quad 0 \leq k \leq p, \tag{30}$$

$$\begin{aligned} \Delta_k^{1-k} &= \frac{1}{k!}[\delta,[\delta,\ldots,[\delta,b]\ldots]], \\ &= \frac{1}{k!}\sum_{r=0}^{k-2}(-1)^r\frac{(k-2)!}{(k-2-r)!r!}\delta^{k-2-r}[\delta,d]\delta^r. \end{aligned} \tag{31}$$

Taking $k = 1$ in (31) gives $d = [\delta, b]$. This condition should be implemented over each field. Indeed, when applied over $\varphi_i^{1-i}, \eta_j^{2-j}, 0 \leq i \leq q, 0 \leq j \leq p$ we obtain the δ-transformation of $d\varphi_i^{i-1}, d\eta_j^{2-j}, 0 \leq i \leq (q-1), 0 \leq j \leq (p-1)$ as

$$\delta d\varphi_k^{1-k} = (k+1)d\varphi_{k+1}^{-k}, \quad 0 \leq k \leq q-2, \tag{32}$$

$$\delta d\varphi_{q-1}^{2-q} = -d\varphi_q^{1-q} - \frac{q+1}{2}\sum_{i=1}^{q}[\varphi_i^{1-i}, \varphi_{q+1-i}^{-q+i}] + (q+1)\eta_{q+1}^{-q+1}, \tag{33}$$

$$\delta d\eta_k^{2-k} = (k+1)d\eta_{k+1}^{-k+1}, \quad 0 \leq k \leq p-2, \tag{34}$$

$$\delta d\eta_{p-1}^{3-p} = -d\eta_p^{2-p} - (p+1)\sum_{i=1}^{p+1}[\varphi_i^{1-i}, \eta_{p+1-i}^{-p+1-i}]. \tag{35}$$

Acting $d = [\delta, b]$ on $d\varphi_q^{1-q}$ and using $p \leq q$ we are let with

$$\begin{aligned} b\delta d\varphi_q^{1-q} = {} & [\delta d\varphi_q^{1-q}, \varphi_0^1] - \delta d\eta_q^{2-q} + (q+1)\sum_{1=1}^{q-1}[d\varphi_i^{1-i}, \varphi_{q+1-i}^{-q+i}] \\ & - \frac{q+1}{2}\sum_{i=1}^{q}[[\varphi_i^{1-i}, \varphi_{q+1-i}^{-q+i}], \varphi_1^0]. \end{aligned} \tag{36}$$

In order to solve this equation we observe that $\delta d\varphi_q^{1-q}$ is a field of bidegree $(q+2, -q)$, therefore it can be written as $\delta d\varphi_q^{1-q} = \alpha \sum_{i=2}^{q}[\varphi_i^{1-i}, \varphi_{q+2-i}^{-q-1+i}]$ which substituting on (36) fixes $\alpha = -(q+1)/2$ and reduces the last equation to

$$\delta d\eta_q^{2-q} = (q+1)\sum_{i=2}^{q}[\eta_i^{2-i}, \varphi_{q+2-i}^{-q-1+i}]. \tag{37}$$

If we suppose that $p < q$ we have $\delta d\eta_q^{2-q} = 0$ and this gives $\sum_{i=2}^{q}[\eta_i^{2-i}, \varphi_{q+2-i}^{-q-1+i}] = 0$ which introduces an unwanted constraint on the fields. Therefore, we should consider $p = q$. Eq. (37) then determines $\delta d\eta_q^{2-q}$. It is straightforward to show that applying $d = [\delta, b]$ on $d\eta_q^{2-q}$ we will obtain the same equation for $\delta d\eta_q^{2-q}$. We can also avoid the previous constraint by setting $\eta_i^{2-i} = 0$, which corresponds to take $p = 0$. These are the zero curvature models of Section 4.

Once we have determined the action of δ on the fields and their derivatives we have fixed the form of the operators Δ_i^{1-i}. It is straightforward to show that the consistency equations for $\Delta_i^{1-i}, i \geq 2$:

$$\sum_{i=2}^{k}\Delta_i^{1-i}\varphi_{k-i}^{1-k+i} = 0, \quad 2 \leq k \leq q \leq D, \tag{38}$$

$$\sum_{i=2}^{k}\Delta_i^{1-i}\varphi_{k-i}^{1-k+i} = -d\varphi_{k-1}^{2-k} - \frac{1}{2}\sum_{i=1}^{k-1}[\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}], \quad k = q+1, \tag{39}$$

$$\sum_{i=k-q}^{k} \Delta_i^{1-i}\varphi_{k-i}^{1-k+i} = -\frac{1}{2}\sum_{i=k-q}^{q}[\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}], \quad q+2 \leq k \leq D, \tag{40}$$

$$\sum_{i=2}^{k} \Delta_i^{1-i}\eta_{k-i}^{2-k+i} = 0, \quad 2 \leq k \leq q \leq D, \tag{41}$$

$$\sum_{i=2}^{k} \Delta_i^{1-i}\eta_{k-i}^{2-k+i} = -d\eta_{k-1}^{3-k} - \sum_{i=1}^{k-1}[\varphi_i^{1-i}, \eta_{k-i}^{2-k+i}], \quad k = q+1, \tag{42}$$

$$\sum_{i=k-q}^{k} \Delta_i^{1-i}\eta_{k-i}^{2-k+i} = -\sum_{i=k-q}^{q}[\varphi_i^{1-i}, \eta_{k-i}^{2-k+i}], \quad q+2 \leq k \leq D \tag{43}$$

are satisfied for this choice of Δ_i^{1-i}.

For convenience we collect below all transformations of our model (with $p = q$):

$$b\varphi_k^{1-k} = -d\varphi_{k-1}^{2-k} - \frac{1}{2}\sum_{i=0}^{k}[\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}] + \eta_k^{2-k}, \quad 0 \leq k \leq q, \tag{44}$$

$$b\eta_k^{2-k} = -d\eta_{k-1}^{3-k} - \sum_{i=0}^{k}[\varphi_i^{1-i}, \eta_{k-i}^{2-k+i}], \quad 0 \leq k \leq q, \tag{45}$$

$$bd\varphi_k^{1-k} = \sum_{i=0}^{k}[d\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}] - d\eta_k^{2-k}, \quad 0 \leq k \leq q, \tag{46}$$

$$bd\eta_k^{2-k} = \sum_{i=0}^{k}[d\varphi_i^{1-i}, \eta_{k-i}^{2-k+i}] - \sum_{i=0}^{k}[\varphi_i^{1-i}, d\eta_{k-i}^{2-k+i}], \quad 0 \leq k \leq q, \tag{47}$$

$$\delta\varphi_k^{1-k} = (k+1)\varphi_{k+1}^{-k}, \quad 0 \leq k \leq q, \tag{48}$$

$$\delta d\varphi_k^{1-k} = (k+1)d\varphi_{k+1}^{-k}, \quad 0 \leq k \leq q-2, \tag{49}$$

$$\delta d\varphi_{q-1}^{2-q} = -d\varphi_q^{1-q} - \frac{q+1}{2}\sum_{i=1}^{q}[\varphi_i^{1-i}, \varphi_{q+1-i}^{-q+i}], \tag{50}$$

$$\delta d\varphi_q^{1-q} = -\frac{q+1}{2}\sum_{i=2}^{q}[\varphi_i^{1-i}, \varphi_{q+2-i}^{-q-1+i}], \tag{51}$$

$$\delta\eta_k^{2-k} = (k+1)\eta_{k+1}^{-k+1}, \quad 0 \leq k \leq q, \tag{52}$$

$$\delta d\eta_k^{2-k} = (k+1) d\eta_{k+1}^{-k}, \quad 0 \le k \le q-2, \tag{53}$$

$$\delta d\eta_{q-1}^{3-q} = -d\eta_q^{2-q} - (q+1) \sum_{i=1}^{q} [\varphi_i^{1-i}, \eta_{q+1-i}^{-q+1-i}], \tag{54}$$

$$\delta d\eta_q^{2-q} = -(q+1) \sum_{i=2}^{q} [\varphi_i^{1-i}, \eta_{q+2-i}^{-q+i}]. \tag{55}$$

It is important to notice that in the case $q = D$, Eqs. (50), (51), (54) and (55) vanish trivially. In this case, all δ transformations of the field derivatives are encoded on (49) and (53) that essentially mean $[\delta, d] = 0$. Then, from (31) we have $\Delta_i^{1-i} = 0$, $i \ge 2$ and consequently all consistency equations (38)–(43) will vanish.

3. A model with $q = 2$, $D = 4$

Let

$$\mathcal{W} = c + A + \varphi_2^{-1}, \tag{56}$$

$$\mathcal{F} = \phi + \psi + B, \tag{57}$$

$$\tilde{d} = b + d + \Delta_2^{-1} + \Delta_3^{-2} + \Delta_4^{-3}. \tag{58}$$

The BRST transformations corresponding to the generalized connection and curvature equations (3)–(5) are given by

$$bc = -c^2 + \phi, \tag{59}$$

$$bA = -dc - [c, A] + \psi, \tag{60}$$

$$b\varphi_2^{-1} = -F - [c, \varphi_2^{-1}] + B, \tag{61}$$

$$b\phi = -[c, \phi], \tag{62}$$

$$b\psi = -d\phi - [c, \psi] - [A, \phi], \tag{63}$$

$$bB = -d\psi - [c, B] - [A, \psi] - [\varphi_2^{-1}, \phi], \tag{64}$$

the δ transformations have the form:

$$\delta c = A, \qquad \delta dc = dA, \tag{65}$$

$$\delta A = 2\varphi_2^{-1}, \qquad \delta dA = -d\varphi_2^{-1} - 3[A, \varphi_2^{-1}], \tag{66}$$

$$\delta\varphi_2^{-1} = 0, \qquad \delta d\varphi_2^{-1} = -3\varphi_2^{-1}\varphi_2^{-1}, \tag{67}$$

$$\delta\phi = \psi, \qquad \delta d\phi = d\psi, \tag{68}$$

$$\delta\psi = 2B, \qquad \delta d\psi = -dB - 3[A, B] - 3[\varphi_2^{-1}, \psi], \tag{69}$$

$$\delta B = 0, \qquad \delta dB = -3[\varphi_2^{-1}, B] \tag{70}$$

and the Δ transformations are given by

$$\Delta_2^{-1} c = 0 \tag{71}$$

$$\Delta_2^{-1} A = -\tfrac{3}{2} d\varphi_2^{-1} - \tfrac{3}{2}[A, \varphi_2^{-1}], \tag{72}$$

$$\Delta_2^{-1} \varphi_2^{-1} = -\tfrac{3}{2}\varphi_2^{-1}\varphi_2^{-1}, \tag{73}$$

$$\Delta_2^{-1} \phi = 0, \tag{74}$$

$$\Delta_2^{-1} \psi = -\tfrac{3}{2} dB - \tfrac{3}{2}[A, B] - \tfrac{3}{2}[\varphi_2^{-1}, \psi], \tag{75}$$

$$\Delta_2^{-1} B = -\tfrac{3}{2}[\varphi_2^{-1}, B], \tag{76}$$

$$\Delta_2^{-1} dc = 0, \tag{77}$$

$$\Delta_2^{-1} dA = -\tfrac{3}{2}[\varphi_2^{-1}, dA] - \tfrac{3}{2}[A, d\varphi_2^{-1}], \tag{78}$$

$$\Delta_2^{-1} d\varphi_2^{-1} = -\tfrac{3}{2}[\varphi_2^{-1}, d\varphi_2^{-1}], \tag{79}$$

$$\Delta_2^{-1} d\phi = 0, \tag{80}$$

$$\Delta_2^{-1} d\psi = -\tfrac{3}{2}[B, dA] - \tfrac{3}{2}[A, dB] - \tfrac{3}{2}[\psi, d\varphi_2^{-1}] - \tfrac{3}{2}[\varphi_2^{-1}, d\psi], \tag{81}$$

$$\Delta_2^{-1} dB = -\tfrac{3}{2}[B, d\varphi_2^{-1}] - \tfrac{3}{2}[\varphi_2^{-1}, dB], \tag{82}$$

$$\Delta_3^{-2} c = \tfrac{1}{2} d\varphi_2^{-1} + \tfrac{1}{2}[A, \varphi_2^{-1}], \tag{83}$$

$$\Delta_3^{-2} A = \tfrac{1}{2}\varphi_2^{-1}\varphi_2^{-1}, \tag{84}$$

$$\Delta_3^{-2} \varphi_2^{-1} = 0, \tag{85}$$

$$\Delta_3^{-2} \phi = \tfrac{1}{2} dB + \tfrac{1}{2}[A, B] + \tfrac{1}{2}[\varphi_2^{-1}, \psi], \tag{86}$$

$$\Delta_3^{-2} \psi = \tfrac{1}{2}[\varphi_2^{-1}, B], \tag{87}$$

$$\Delta_3^{-2} B = 0, \tag{88}$$

$$\Delta_3^{-2} dc = \tfrac{1}{2}[\varphi_2^{-1}, dA] + \tfrac{1}{2}[A, d\varphi_2^{-1}], \tag{89}$$

$$\Delta_3^{-2} dA = \tfrac{1}{2}[\varphi_2^{-1}, d\varphi_2^{-1}], \tag{90}$$

$$\Delta_3^{-2} d\varphi_2^{-1} = 0, \tag{91}$$

$$\Delta_3^{-2} d\phi = \tfrac{1}{2}[B, dA] + \tfrac{1}{2}[A, dB] + \tfrac{1}{2}[\psi, d\varphi_2^{-1}] + \tfrac{1}{2}[\varphi_2^{-1}, d\psi], \tag{92}$$

$$\Delta_3^{-2} d\psi = \tfrac{1}{2}[B, d\varphi_2^{-1}] + \tfrac{1}{2}[\varphi_2^{-1}, dB], \tag{93}$$

$$\Delta_3^{-2} dB = 0, \tag{94}$$

$$\Delta_4^{-3} \equiv 0. \tag{95}$$

Let us consider now the system of descent equations given in (6). We can rewrite it in the form $(b+d)\tilde{\omega} \equiv (\tilde{d} - \Delta)\tilde{\omega} = 0$ with $\tilde{\omega} \doteq \omega_0^4 + \omega_1^3 + \omega_2^2 + \omega_3^1 + \omega_4^0$ and $\Delta \doteq \Delta_2^{-1} + \Delta_3^{-2} + \Delta_4^{-3}$. A particular solution is given by

$$\tilde{\omega} = \mathrm{e}^{\delta}(\omega_0^4 + \Omega) \tag{96}$$

with $\Omega \doteq \Omega_1^3 + \Omega_2^2 + \Omega_3^1 + \Omega_4^0$ satisfying

$$b\Omega_4^0 = \Delta_2^{-1}\Omega_2^2 - 2\Delta_3^{-2}\Omega_1^3 + 3\Delta_4^{-3}\omega_0^4, \tag{97}$$

$$b\Omega_3^1 = \Delta_2^{-1}\Omega_1^3 - 2\Delta_3^{-2}\omega_0^4, \tag{98}$$

$$b\Omega_2^2 = \Delta_2^{-1}\omega_0^4, \tag{99}$$

$$b\Omega_1^3 = 0. \tag{100}$$

In terms of these Ω's we have

$$\omega_4^0 = \frac{\delta^4}{4!}\omega_0^4 + \frac{\delta^3}{3!}\omega_1^3 + \frac{\delta^2}{2!}\Omega_2^2 + \delta\Omega_3^1 + \Omega_4^0, \tag{101}$$

$$\omega_3^1 = \frac{\delta^3}{3!}\omega_0^4 + \frac{\delta^2}{2!}\Omega_1^3 + \delta\Omega_2^2 + \Omega_3^1, \tag{102}$$

$$\omega_2^2 = \frac{\delta^2}{2!}\omega_0^4 + \delta\Omega_1^3 + \Omega_2^2, \tag{103}$$

$$\omega_1^3 = \delta\omega_0^4 + \Omega_1^3. \tag{104}$$

Here we notice that the cycles exhibited in (101)–(104) are obtained from Ω's by the action of δ. These Ω's are solutions of the intermediate equations (97)–(100), which do not involve the exterior derivative. It is the combination of the δ-operator and these equations ((97)–(100)) that allow us to transform a problem of cohomology of b modulo d (6) into a simple one. In order to solve (101)–(104) we should first determine ω_0^4, the solution of $b\omega_0^4 = 0$. Our intention is to analyze how the cocycle $\mathrm{Tr}(1/2)\phi^2$ (which appears in [1,3]) is modified by the presence of the negative ghost number field φ_2^{-1}, the field B, and the operators Δ_2^{-1}, Δ_3^{-2}, Δ_4^{-3}. Therefore we take

$$\omega_0^4 = \mathrm{Tr}(\tfrac{1}{2}\phi^2). \tag{105}$$

Then, we obtain Ω's solving (97)–(100). Replacing them in (101)–(104) we obtain

$$\begin{aligned}\omega_1^3 = \mathrm{Tr}\{&\tfrac{2}{3}\beta_1(c^2\psi - c^2dc + c[A,\phi]) + \tfrac{2}{3}(\beta_2 - \beta_4)(\phi\psi - \phi dc)\\ &+ \tfrac{2}{3}\beta_3(-c^2\psi + c^2dc - \phi\psi + \phi dc - c[A,\phi])\\ &+ \sigma(c^2dc + cd\phi + \phi dc) + \tfrac{1}{3}\phi\psi + \tfrac{2}{3}\phi dc\},\end{aligned} \tag{106}$$

$$
\begin{aligned}
\omega_2^2 = \mathrm{Tr}\{&\tfrac{2}{3}\beta_1(2c^2B - c^2dA + 2A^2\phi + 2c[A,\psi] - c[A,dc] + 2c[\varphi_2^{-1},\phi] \\
&+ \tfrac{2}{3}(\beta_2-\beta_4)(2\phi B - \phi dA + \psi^2 - \psi dc) + \tfrac{2}{3}\beta_3(-2c^2B + c^2dA - 2A^2\phi \\
&- 2\phi B + \phi dA - \psi^2 + \psi dc - 2c[A,\psi] + c[A,dc] - 2c[\varphi_2^{-1},\phi]) \\
&+ \alpha_1(c^2A^2 - c^2B + c^2dA - c[\varphi_2^{-1},\phi]) + \alpha_2(A^2\phi - \phi B + \phi dA) \\
&+ \alpha_3(c^2B + cd\psi + \phi B + c[A,\psi] + c[\varphi_2^{-1},\phi]) \\
&+ \alpha_4(-c^2dA - cd\psi - \phi dA - c[A,dc]) + \alpha_5(Ad\phi + \tfrac{1}{2}\psi^2 + \phi B - \phi dA - \tfrac{1}{2}dcdc) \\
&+ \alpha_6(-\tfrac{1}{2}\psi^2 + \psi dc - \phi B + \phi dA - \tfrac{1}{2}dcdc) + \sigma(c^2dA + cd\psi + Ad\phi \\
&+ \phi dA + \psi dc + c[A,dc]) - \tfrac{1}{3}\phi B + \tfrac{2}{3}\phi dA - \tfrac{1}{6}\psi^2 + \tfrac{2}{3}\psi dc\},
\end{aligned}
\tag{107}
$$

$$
\begin{aligned}
\omega_3^1 = \mathrm{Tr}\{&2\beta_1(\tfrac{2}{3}c^2d\varphi_2^{-1} + A^2\psi - \tfrac{1}{3}A^2dc + c^2[A,\varphi_2^{-1}] + c[A,B] - \tfrac{1}{3}c[A,dA] \\
&+ c[\varphi_2^{-1},\psi] - \tfrac{1}{3}c[\varphi_2^{-1},dc] + A[\varphi_2^{-1},\phi]) + 2\beta_2(\tfrac{2}{3}\phi d\varphi_2^{-1} + \psi B - \tfrac{1}{3}\psi dA \\
&- \tfrac{1}{3}Bdc + A[\varphi_2^{-1},\phi]) + \beta_3(-\tfrac{1}{3}c^2d\varphi_2^{-1} + cdB - 2A^2\psi + \tfrac{2}{3}A^2dc - \tfrac{1}{3}\phi d\varphi_2^{-1} \\
&- 2\psi B + \tfrac{2}{3}\psi dA + \tfrac{2}{3}Bdc - c^2[A,\varphi_2^{-1}] - c[A,B] + \tfrac{2}{3}c[A,dA] - c[\varphi_2^{-1},\psi] \\
&+ \tfrac{2}{3}c[\varphi_2^{-1},dc] - 3A[\varphi_2^{-1},\phi]) + \beta_4(-\tfrac{4}{3}\phi d\varphi_2^{-1} - \psi B - \tfrac{1}{3}\psi dA - \tfrac{1}{3}Bdc \\
&+ dcdA - A[\varphi_2^{-1},\phi]) + \beta_5(c^2\varphi_2^{-1}A + cA^3 + c\varphi_2^{-1}\psi - c\varphi_2^{-1}dc - cBA \\
&+ cdAA - A\phi\varphi_2^{-1}) + \beta_6(A^2dc - \varphi_2^{-1}d\phi - Bdc + dcdA) \\
&+ \beta_7(A^2\psi - \varphi_2^{-1}d\phi - Bdc + dcdA) + \beta_8(c^2d\varphi_2^{-1} + cdB + \phi d\varphi_2^{-1} + c[A,dA] \\
&+ c[\varphi_2^{-1},dc]) + \beta_9(Ad\psi + 2\varphi_2^{-1}d\phi + \psi dA + 2Bdc - 3dcdA) \\
&+ \alpha_1(-c^2d\varphi_2^{-1} - c^2[A,\varphi_2^{-1}] - c[A,B] + c[A,dA] - c[\varphi_2^{-1},\psi] - A[\varphi_2^{-1},\phi]) \\
&+ \alpha_2(A^2\psi - \phi d\varphi_2^{-1} - \psi B + \psi dA - A[\varphi_2^{-1},\phi]) + \alpha_3(-cdB + 2A^2\psi \\
&+ Ad\psi + \psi B + A[\varphi_2^{-1},\phi]) + \alpha_4(c^2d\varphi_2^{-1} + cdB - 2A^2dc - Ad\psi \\
&+ \phi d\varphi_2^{-1} - \psi dA + 3c^2[A,\varphi_2^{-1}] + 3c[A,B] - 2c[A,dA] + 3A[\varphi_2^{-1},\phi] \\
&+ 3c[\varphi_2^{-1},\psi] - 2c[\varphi_2^{-1},dc]) + \alpha_5(Ad\psi + 2\varphi_2^{-1}d\phi + \phi d\varphi_2^{-1} + 3\psi B - \psi dA \\
&- dcdA + 3A[\varphi_2^{-1},\phi]) + \alpha_6(-\phi d\varphi_2^{-1} - 3\psi B + 2\psi dA + 2Bdc - dcdA \\
&- 3A[\varphi_2^{-1},\phi]) + \sigma(-\tfrac{1}{2}c^2d\varphi_2^{-1} - \tfrac{1}{2}cdB + A^2dc + Ad\psi + \varphi_2^{-1}d\phi - \tfrac{1}{2}\phi d\varphi_2^{-1} \\
&+ \psi dA + Bdc - \tfrac{3}{2}c^2[A,\varphi_2^{-1}] - \tfrac{3}{2}c[A,B] + c[A,dA] - \tfrac{3}{2}c[\varphi_2^{-1},\psi] \\
&+ c[\varphi_2^{-1},dc] - \tfrac{3}{2}A[\varphi_2^{-1},\phi]) - \tfrac{1}{3}\phi d\varphi_2^{-1} - \psi B + \tfrac{2}{3}\psi dA + \tfrac{2}{3}Bdc - A[\varphi_2^{-1},\phi]\},
\end{aligned}
\tag{108}
$$

$$
\begin{aligned}
\omega_4^0 = \mathrm{Tr}\{&2\beta_1(c^2\varphi_2^{-1}\varphi_2^{-1} + \tfrac{4}{3}A^2B - \tfrac{1}{3}A^2dA + \varphi_2^{-1}\varphi_2^{-1}\phi + c[A^2,\varphi_2^{-1}] + c[A,d\varphi_2^{-1}] \\
&+ c[\varphi_2^{-1},B] + \tfrac{4}{3}A[\varphi_2^{-1},\psi] - \tfrac{1}{3}A[\varphi_2^{-1},dc]) + 2\beta_2(\tfrac{4}{3}\varphi_2^{-1}\varphi_2^{-1}\phi + \psi d\varphi_2^{-1} \\
&+ \tfrac{2}{3}B^2 - \tfrac{1}{3}BdA - \tfrac{1}{3}dcd\varphi_2^{-1} + \tfrac{4}{3}A[\varphi_2^{-1},\psi] - \tfrac{1}{3}A[\varphi_2^{-1},dc]) \\
&+ \beta_3(-c^2\varphi_2^{-1}\varphi_2^{-1} - \tfrac{2}{3}A^2B + \tfrac{2}{3}A^2dA + AdB - \tfrac{11}{3}\varphi_2^{-1}\varphi_2^{-1}\phi - \psi d\varphi_2^{-1} - \tfrac{4}{3}B^2 \\
&+ \tfrac{2}{3}BdA + \tfrac{2}{3}dcd\varphi_2^{-1} - c[A^2,\varphi_2^{-1}] - c[A,d\varphi_2^{-1}] - c[\varphi_2^{-1},B] - \tfrac{10}{3}A[\varphi_2^{-1},\psi]
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{4}{3}A[\varphi_2^{-1},dc])+\beta_4(\tfrac{4}{3}\varphi_2^{-1}\varphi_2^{-1}\phi-\psi d\varphi_2^{-1}+\tfrac{2}{3}B^2-\tfrac{7}{3}BdA-\tfrac{1}{3}dcd\varphi_2^{-1}\\
&+dAdA+\tfrac{4}{3}A[\varphi_2^{-1},\psi]-\tfrac{7}{3}A[\varphi_2^{-1},dc])+\beta_5(2c^2\varphi_2^{-1}\varphi_2^{-1}-c\varphi_2^{-1}dA\\
&+2cdA\varphi_2^{-1}-cd\varphi_2^{-1}A+A^4+A^2B-A^2dA-A\varphi_2^{-1}dc+2\varphi_2^{-1}\varphi_2^{-1}\phi\\
&+2c[A^2,\varphi_2^{-1}]+2c[\varphi_2^{-1},B]+A[\varphi_2^{-1},\psi])+\beta_6(A^2dA-\varphi_2^{-1}d\psi-BdA\\
&-dcd\varphi_2^{-1}+dAdA-A[\varphi_2^{-1},dc])+\beta_7(2A^2B-\varphi_2^{-1}d\psi-BdA-dcd\varphi_2^{-1}\\
&+dAdA+2A[\varphi_2^{-1},\psi]-3A[\varphi_2^{-1},dc])+\beta_8(-3c^2\varphi_2^{-1}\varphi_2^{-1}+2A^2dA\\
&+AdB-3\varphi_2^{-1}\varphi_2^{-1}\phi+\psi d\varphi_2^{-1}-3c[A^2,\varphi_2^{-1}]-3c[\varphi_2^{-1},B]+3c[\varphi_2^{-1},dA]\\
&+A[\varphi_2^{-1},dc])+\beta_9(-6A^2B-AdB+4\varphi_2^{-1}d\psi-\psi d\varphi_2^{-1}+4BdA\\
&+3dcd\varphi_2^{-1}-3dAdA-6A[\varphi_2^{-1},\psi]+9A[\varphi_2^{-1},dc])+\alpha_1(-2c^2\varphi_2^{-1}\varphi_2^{-1}\\
&-A^2B+A^2dA-2\varphi_2^{-1}\varphi_2^{-1}\phi-2c[A^2,\varphi_2^{-1}]-c[A,d\varphi_2^{-1}]-2c[\varphi_2^{-1},B]\\
&+c[\varphi_2^{-1},dA]-A[\varphi_2^{-1},\psi])+\alpha_2(A^2B-2\varphi_2^{-1}\varphi_2^{-1}\phi-\psi d\varphi_2^{-1}-B^2+BdA\\
&-A[\varphi_2^{-1},\psi])+\alpha_3(-A^2B-AdB+2\varphi_2^{-1}\varphi_2^{-1}\phi+\varphi_2^{-1}d\psi+B^2+A[\varphi_2^{-1},\psi])\\
&+\alpha_4(\tfrac{9}{2}c^2\varphi_2^{-1}\varphi_2^{-1}+6A^2B-3A^2dA+AdB+\tfrac{9}{2}\varphi_2^{-1}\varphi_2^{-1}\phi-\varphi_2^{-1}d\psi+\psi d\varphi_2^{-1}\\
&-BdA+\tfrac{9}{2}c[A^2,\varphi_2^{-1}]+3c[A,d\varphi_2^{-1}]+\tfrac{9}{2}c[\varphi_2^{-1},B]-\tfrac{3}{2}c[\varphi_2^{-1},dA]\\
&+6A[\varphi_2^{-1},\psi]-3A[\varphi_2^{-1},dc])+\alpha_5(-3A^2B-\tfrac{1}{2}AdB+6\varphi_2^{-1}\varphi_2^{-1}\phi\\
&+2\varphi_2^{-1}d\psi+\tfrac{5}{2}\psi d\varphi_2^{-1}+3B^2-BdA+\tfrac{1}{2}dcd\varphi_2^{-1}-\tfrac{1}{2}dAdA+3A[\varphi_2^{-1},\psi]\\
&+\tfrac{3}{2}A[\varphi_2^{-1},dc])+\alpha_6(-6\varphi_2^{-1}\varphi_2^{-1}\phi-3\psi d\varphi_2^{-1}-3B^2+3BdA+2dcd\varphi_2^{-1}\\
&-\tfrac{1}{2}dAdA-6A[\varphi_2^{-1},\psi]+3A[\varphi_2^{-1},dc])+\gamma_1(c^2\varphi_2^{-1}\varphi_2^{-1}+\varphi_2^{-1}\varphi_2^{-1}\phi\\
&+c[A^2,\varphi_2^{-1}]+c[\varphi_2^{-1},B]-c[\varphi_2^{-1},dA])+\gamma_2(-A^2B+A^2dA\\
&-A[\varphi_2^{-1},\psi]+A[\varphi_2^{-1},dc])+\gamma_3(2A^2B+AdB+\psi d\varphi_2^{-1}-dcd\varphi_2^{-1}\\
&+2A[\varphi_2^{-1},\psi]-A[\varphi_2^{-1},dc])+\gamma_4(-A^2B+2\varphi_2^{-1}\varphi_2^{-1}\phi+\varphi_2^{-1}d\psi+B^2\\
&-BdA+A[\varphi_2^{-1},\psi])+\gamma_5(A^2B-\varphi_2^{-1}d\psi-BdA+dAdA+A[\varphi_2^{-1},\psi]\\
&-2A[\varphi_2^{-1},dc])+\sigma(-\tfrac{3}{2}c^2\varphi_2^{-1}\varphi_2^{-1}-3A^2B+A^2dA-\tfrac{1}{2}AdB-\tfrac{3}{2}\varphi_2^{-1}\varphi_2^{-1}\phi\\
&+\varphi_2^{-1}d\psi-\tfrac{1}{2}\psi d\varphi_2^{-1}+BdA+dcd\varphi_2^{-1}-\tfrac{3}{2}c[A^2,\varphi_2^{-1}]-\tfrac{3}{2}c[A,d\varphi_2^{-1}]\\
&-\tfrac{3}{2}c[\varphi_2^{-1},B]-3A[\varphi_2^{-1},\psi]+2A[\varphi_2^{-1},dc])-\tfrac{5}{3}\varphi_2^{-1}\varphi_2^{-1}\phi-\psi d\varphi_2^{-1}-\tfrac{5}{6}B^2\\
&+\tfrac{2}{3}BdA+\tfrac{2}{3}dcd\varphi_2^{-1}-\tfrac{5}{3}A[\varphi_2^{-1},\psi]+\tfrac{2}{3}A[\varphi_2^{-1},dc]\}. \qquad (109)
\end{aligned}
$$

Considered in this form, this previous solution for ω_i^{1-i} does not relate to any familiar model. Here, let us consider some specific cases. First, let us consider the two-form B decomposing as $B=F+\hat{B}$ [5] with F the curvature of A. In this decomposition, the two-form $\hat{B}$ should be introduced in order to maintain the nilpotency of the BRST transformation of φ_2^{-1}. We have $b^2\varphi_2^{-1}=0\Rightarrow b\hat{B}=-[c,\hat{B}]+[\phi,\varphi_2^{-1}]$. Then, taking $\beta_2=1$ with all other parameters set to zero we obtain ω_i^{1-i} as

$$\omega_0^4=\mathrm{Tr}(\tfrac{1}{2}\phi^2), \qquad (110)$$

$$\omega_1^3 = \mathrm{Tr}(\phi\psi), \tag{111}$$

$$\omega_2^2 = \mathrm{Tr}(\phi F + \tfrac{1}{2}\psi^2 + \phi\hat{B}), \tag{112}$$

$$\omega_3^1 = \mathrm{Tr}(\psi F + \phi D_A\varphi_2^{-1} + \psi\hat{B}), \tag{113}$$

$$\omega_4^0 = \mathrm{Tr}(\tfrac{1}{2}F^2 + \tfrac{1}{2}\hat{B}^2 + F\hat{B} + \varphi_2^{-1}\varphi_2^{-1}\phi + \psi D_A\varphi_2^{-1}). \tag{114}$$

We observe that the inclusion of additional fields φ_2^{-1}, B in the ladders, and of additional derivations Δ_i^{1-i} in $\tilde{d}$ modify the previous solution (7) of the descent equations. Nonetheless, (110)–(114) still *contains* the terms associated to the Donaldson polynomials. A similar behavior has been observed in [5] for the case $\varphi_2^{-1} = 0$, $B \neq F$, which also generates a solution including additional terms to the Donaldson polynomials. Now, if we look at our general solution (106)–(109) we see that they represent a family of solutions parameterized by 21 parameters $(\beta_1, \dots, \sigma)$ which writes as $\tilde{\omega} = (1/2)(\phi+\psi+F)^2 + ((1/2)\hat{B}^2 + \phi\hat{B} + \psi\hat{B} + F\hat{B} + \phi D_A\varphi_2^{-1} + \psi D_A\varphi_2^{-1} + \varphi_2^{-1}\varphi_2^{-1}\phi) + \Theta(\beta_1, \dots, \sigma)$. Here, there is no possibility to choose the parameters $(\beta_1, \dots, \sigma)$ in such a way that $\tilde{\omega}$ reduces to the Donaldson polynomials. From [5], it seems then that the only cases having a complete agreement with (7) are $\varphi_2^{-1} = 0$, $B = F$ that gives the same result as (7), and $\varphi_2^{-1} = 0$, $B = 0$ that represents a family of solutions parameterized by points of R^8 and such that to the origin we have associated (7), i.e. $\tilde{\omega} = (1/2)(\phi+\psi+F)^2 + \Theta(\alpha_1, \dots, \alpha_8)$ with $\tilde{\omega}|_{(\alpha_1,\dots,\alpha_8)=0} = (1/2)(\phi+\psi+F)^2$. This solution is interesting because it shows Donaldson generators as a particular case of a more general expression. Therefore, it may be possible that other extended formulations may admit, as a limit case, other topological invariants. Nonetheless, up to the analysis of this example, it is not known if a choice of higher components ladders would generate a solution of this type.

The cycle ω_4^0 is particularly important since it defines a BRST invariant action:

$$\mathcal{S} = \int \mathrm{Tr}\left(\frac{1}{2}F^2 + \frac{1}{2}\hat{B}^2 + F\hat{B} + \varphi_2^{-1}\varphi_2^{-1}\phi + \psi D_A\varphi_2^{-1}\right), \tag{115}$$

which can be taken as the starting point for a pertubative analysis of our model. This action incorporates, from the beginning, extra terms on φ_2^{-1}, $\hat{B}$ in addition to the usual non-gauge fixed TYMT action $\int \mathrm{Tr}\, F^2$. Thus, in much the same way as it was done in [24], we may interpret the fields φ_2^{-1}, $\hat{B}$ as part of the additional fields necessary to perform the gauge fixing of the action $\int \mathrm{Tr}\, F^2$. If we want to proceed further on finding a fully gauge fixed action, we will have to introduce other fields (antifields, antighosts) with total degree different than 0 and 1, which will be accommodated as component fields of other ladders.

Another application of the model given by (56)–(58) is on the description of four-dimensional BF model. In fact, consider the cycle ω_4^0 (109). Let us take $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 1/3$, $\gamma_4 = -(1/2)$, $\gamma_5 = -1/6$, $\beta_2 = 1$ with all other parameters equal to zero. Then, we obtain an invariant action given by[3]

[3] Here, we may also interpret φ_2^{-1} as one of the fields necessary to perform the gauge fixing of the BF action.

$$S = \int \omega_4^0 = \int \mathrm{Tr}(BF + \psi D_A \varphi_2^{-1} + \varphi_2^{-1}\varphi_2^{-1}\phi + B[c, \varphi_2^{-1}] \\ + c^2\varphi_2^{-1}\varphi_2^{-1} - c[\varphi_2^{-1}, F]), \tag{116}$$

which contains the usual term of the BF model. It is important to notice that this derivation of four-dimensional BF action is based on a pair of connection and curvature ladders (56) and (57) with the assumption that $B \neq F$. In contrast, the usual superfield formulation of D-dimensional BF models [6,25] employs a gauge ladder together with a matter ladder $\mathcal{B}$ having the two-form B as its highest component field, i.e. $\mathcal{B}|_D = B$. In Section 4.2 we will obtain the equivalent of action (116) for the zero curvature formulation of four-dimensional BF model.

4. The zero-curvature models

As we have seen, the model presented in Section 2 is based on gauge and curvature ladders $\mathcal{W}, \mathcal{F}$ satisfying $\tilde{d}\mathcal{W} + (1/2)[\mathcal{W}, \mathcal{W}] = \mathcal{F}, \tilde{d}\mathcal{F} + [\mathcal{W}, \mathcal{F}] = 0$. As a limit case of this model we can pose a zero curvature condition $\mathcal{F} = 0$ that reduces the previous equations to $\tilde{d}\mathcal{W} + (1/2)[\mathcal{W}, \mathcal{W}] = 0$. Here, (44)–(55) become

$$b\varphi_k^{1-k} = -d\varphi_{k-1}^{2-k} - \frac{1}{2}\sum_{i=0}^{k}[\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}], \quad 0 \le k \le q, \tag{117}$$

$$bd\varphi_k^{1-k} = \sum_{i=0}^{k}[d\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}], \quad 0 \le k \le q, \tag{118}$$

$$\delta\varphi_k^{1-k} = (k+1)\varphi_{k+1}^{-k}, \quad 0 \le k \le q, \tag{119}$$

$$\delta d\varphi_k^{1-k} = (k+1)d\varphi_{k+1}^{-k}, \quad 0 \le k \le q-2, \tag{120}$$

$$\delta d\varphi_{q-1}^{2-q} = -d\varphi_q^{1-q} - \frac{q+1}{2}\sum_{i=1}^{q}[\varphi_i^{1-i}, \varphi_{q+1-i}^{-q+i}], \tag{121}$$

$$\delta d\varphi_q^{1-q} = -\frac{q+1}{2}\sum_{i=2}^{q}[\varphi_i^{1-i}, \varphi_{q+2-i}^{-q-1+i}] \tag{122}$$

that agree with the same equations obtained in the non-complete ladder case (i.e. with $q \neq D$) of [7]. In our approach we treat both cases $q = D$ (referred in [6] as the complete ladder case) and $q \neq D$ in the same way, with the fundamental equations given as above. Indeed, the equations for the complete ladder case are a particular case of (117)–(122) when one takes $q = D$. Basically, what differs one and another situation is just the definition of the generalized derivative that assumes the form $\tilde{d} = b + d$ when $q = D$ and $d = b + d + \sum_{i=2}^{D} \Delta_i^{1-i}$ when $q \neq D$. The main role of the operators Δ_i^{1-i}, $i \ge 2$ is to avoid possible constraints that would arise from the zero curvature condition in the case of $q \neq D$.

For example, in the absence of Δ_i^{1-i} we would have from (39) and (40) the two constraints below

$$d\varphi_q^{1-q} = -\frac{1}{2}\sum_{i=1}^{q}[\varphi_i^{1-i}, \varphi_{q+1-i}^{-q+i}], \qquad 0 = \sum_{i=k-q}^{q}[\varphi_i^{1-i}, \varphi_{k-i}^{1-k+i}], \quad k \geq q+2.$$

As we have pointed out at the end of Section 2, $q = D$ determines $\Delta_i^{1-i} = 0$ and this explains why these operators are absent in the complete ladder case of [6].

Let us consider general descent equations of the type

$$b\omega_{D-i}^{G+i} + d\omega_{D-i-1}^{G+i+1} = 0, \quad 0 \leq i \leq D-1, \qquad b\omega_0^{G+D} = 0. \tag{123}$$

This system of descent equations can be solved following the same procedure of Section 3, e.g. writing $\tilde{\omega} \equiv \sum_{i=0}^{G+D}\omega_i^{G+D}$ and $\Delta \equiv \sum_{i=2}^{D}\Delta_i^{1-i}$ the descent equations assume the form $0 = (b+d)\tilde{\omega} = (\tilde{d} - \Delta)\tilde{\omega}$. A particular solution is $\tilde{\omega} \doteq \mathrm{e}^{\delta}(\omega_0^{G+D} + \Omega)$ with $\Omega \equiv \sum_{i=1}^{D}\Omega_i^{G+D-i}$ satisfying

$$b\Omega_k^{G+D-k} = (-1)^k(k-1)\Delta_k^{1-k}\omega_0^{G+D} + \sum_{i=2}^{k-1}(-1)^i(i-1)\Delta_i^{1-i}\Omega_{k-i}^{G+D-k+i}, \qquad 1 \leq k \leq D. \tag{124}$$

We note that when $q = D$ we have $\Omega = 0$ and $\Delta = 0$, then $\tilde{\omega} = \mathrm{e}^{\delta}\omega_0^{G+D}$ and $\tilde{d} = b + d$. When $G + D = 4$, (124) agrees with (97)–(100).

4.1. The Chern–Simons term

Consider a model with $q = 3$, $D = 3$ and $\mathcal{F} = 0$. Let us take the cocycle ω_3^0 such that $b\int\omega_3^0 = 0$. This will be related to the Chern–Simons form. As it was shown in [6], ω_3^0 can be obtained by expanding $\tilde{\omega} = \mathrm{e}^{\delta}\omega_0^3 = \mathrm{e}^{\delta}((1/3!)\mathrm{Tr}\, c^3)$ and taking the terms with form degree equal to 3. This results in

$$S = \frac{1}{2}\int \mathrm{Tr}\left(AF - \frac{1}{3}A^3\right) - \frac{1}{2}b\int \mathrm{Tr}(c\varphi_3^{-2} + A\varphi_2^{-1}). \tag{125}$$

Nonetheless, the presence of the field φ_2^{-1} allow us to consider a more general solution by introducing on (125) the term

$$\int\left(dc\varphi_2^{-1} + \frac{1}{2}AdA\right). \tag{126}$$

Then, the action given by (125) + (126) is also possible and represents a contribution due to the extra field φ_2^{-1}.

4.2. BF system

The D-dimensional BF system can be formulated as a zero curvature system by introducing two complete ladders [6]:

$$\mathcal{W}=\sum_{i=0}^{D}\varphi_i^{1-i}, \qquad \mathcal{B}=\sum_{i=0}^{D}B_j^{D-2-j}, \tag{127}$$

where $\mathcal{W}$ is a gauge ladder with total degree 1, which satisfies a zero curvature condition. The other ladder $\mathcal{B}$ has total degree $(D-2)$ and satisfies $\tilde{d}\mathcal{B}+[\mathcal{W},\mathcal{B}]=0$. For the complete ladder case, we have seen that $\tilde{d}=b+d$. Let us consider the case $D=4$. Here, the gauge and the matter ladder $\mathcal{B}$ are taken as

$$\mathcal{W}=c+A+\varphi_2^{-1}+\varphi_3^{-2}+\varphi_4^{-3}, \tag{128}$$

$$\mathcal{B}=\phi+\psi+B+B_3^{-1}+B_4^{-2}. \tag{129}$$

The BRST transformations for the component fields follow from the equations satisfied by $\mathcal{W}$ and $\mathcal{B}$ and are given by

$$bc=-c^2, \tag{130}$$

$$bA=-dc-[c,A], \tag{131}$$

$$b\varphi_2^{-1}=-F-[c,\varphi_2^{-1}], \tag{132}$$

$$b\varphi_3^{-1}=-d\varphi_2^{-1}-[c,\varphi_3^{-1}]-[A,\varphi_2^{-1}], \tag{133}$$

$$b\varphi_4^{-3}=-d\varphi_3^{-1}-[c,\varphi_4^{-3}]-[A,\varphi_3^{-2}]-\tfrac{1}{2}[\varphi_2^{-1},\varphi_2^{-1}], \tag{134}$$

$$b\phi=-[c,\phi], \tag{135}$$

$$b\psi=-d\phi-[c,\psi]-[A,\phi], \tag{136}$$

$$bB=-d\psi-[c,B]-[A,\psi]-[\varphi_2^{-1},\phi], \tag{137}$$

$$bB_3^{-1}=-dB-[c,B_3^{-1}]-[A,B]-[\varphi_2^{-1},\psi]-[\varphi_3^{-1},\phi], \tag{138}$$

$$bB_4^{-2}=-dB_3^{-1}-[c,B_4^{-2}]-[A,B_3^{-1}]-[\varphi_2^{-1},B]-[\varphi_3^{-2},\psi]-[\varphi_4^{-3},\phi]. \tag{139}$$

The BRST transformations for ϕ, ψ, B agree with the ones given in (62)–(64). Nonetheless, since $\mathcal{W}$ is a connection with zero curvature, the BRST transformations for the components c, A, φ_2^{-1} differ from (59)–(61).

From [6] we obtain an invariant action as

$$S=\int \mathrm{Tr}\,\mathcal{B}(d\mathcal{W}+\mathcal{W}^2)|_4^0, \tag{140}$$

$$S=\int \mathrm{Tr}(BF+\psi D_A\varphi_2^{-1}+\varphi_2^{-1}\varphi_2^{-1}\phi+B[c,\varphi_2^{-1}]+\phi[c,\varphi_4^{-3}]+\psi[c,\varphi_3^{-2}] \\ +\phi D_A\varphi_3^{-2}+B_3^{-1}D_Ac+B_4^{-2}c^2). \tag{141}$$

This previous action agrees with the one given in (116) except by the presence of higher components fields φ_3^{-2}, φ_4^{-3}, B_3^{-1}, B_4^{-2} that does not enter in the ladders (56) and (57). Conversely, there are also the presence of terms on φ_2^{-1} in (116) that do not appear in (141), those terms being brought by the derivations Δ_i^{1-i}, which are absent on the generalized derivative $\tilde{d} = b + d$. Both approaches are entirely different since they are based on ladders that satisfy different equations. As for the general formulation of BF models in dimensions other than $D = 4$, we emphasize that a matter ladder $\mathcal{B}$, satisfying $\tilde{d}\mathcal{B} + [\mathcal{W}, \mathcal{B}] = 0$, should be used to accommodate the field B. It is a particular feature of four dimensions that we can take the ladder $\mathcal{B}$ (57) as the generalized curvature of $\mathcal{W}$ (56).

5. Mathematical aspects

5.1. BRST $\mathcal{G}$-operation

In this section we review some basic definitions concerning the structure of graded commutative differential algebras and BRST $\mathcal{G}$-operations. Although our approach is based on the formalism exposed in [8,9] we will adopt some definitions in a different context.

A *Z-graded supercommutative algebra* is a structure defined by $(\mathcal{A}, *)$ such that: (1) $(\mathcal{A}, *)$ is an algebra in the usual sense (we are considering algebras defined over a field K that can be R or C), (2) the graded structure is defined by a direct sum decomposition $\mathcal{A} = \oplus_{m\in Z}\mathcal{A}^m$ such that $\mathcal{A}^m * \mathcal{A}^n \subset \mathcal{A}^{m+n}$ and the supercommutativity stands for $\alpha * \beta = (-1)^{mn}\beta * \alpha\ \forall \alpha \in \mathcal{A}^m, \forall \beta \in \mathcal{A}^n$. From now on we will use the term commutative as meaning supercommutative. All graded (bigraded) structure to be considered here will be defined either over Z or $Z^+ \doteq N \cup \{0\}$.

A *superderivation on $\mathcal{A}$ of degree k* is a linear map $\Psi : \mathcal{A}^\bullet \to \mathcal{A}^{\bullet+k}$ such that $\Psi(\alpha\beta) = (\Psi\alpha)\beta + (-1)^{km}\alpha\Psi\beta\ \forall \alpha \in \mathcal{A}^m$. We denote the set of k-superderivations on $\mathcal{A}$ as $\mathcal{D}^k(\mathcal{A})$. Defining a product between two superderivations on $\mathcal{A}$ as the composition map we have that $\mathcal{D}(\mathcal{A}) \equiv \sum_{k\in Z}\mathcal{D}^k(\mathcal{A})$ together with this product becomes a graded algebra.

A *graded commutative differential algebra* is a structure defined by $(\mathcal{A}, *, d)$ with (1) $(\mathcal{A}, *)$ a graded commutative algebra and (2) d a superderivation on $\mathcal{A}$ of degree 1 such that $d^2 = 0$.

A *$\mathcal{G}$-operation* is defined by $(\mathcal{A}, *, d, I, L)$ with (1) $(\mathcal{A}, *, d)$ a graded commutative differential algebra and (2) $I : \mathcal{G} \to \mathcal{D}^{-1}(\mathcal{A})$, $X \to I_X$ and $L : \mathcal{G} \to \mathcal{D}^0(\mathcal{A})$, $X \to L_X \doteq [d, I_X]$ such that $I_{[X,Y]} = L_X I_Y - I_Y L_X$ and $L_{[X,Y]} = L_X L_Y - L_Y L_X\ \forall X, Y \in \mathcal{G}$. We extend these two operations to $\mathcal{G} \otimes \mathcal{A}$ as $I_X(Y \otimes \alpha) \doteq Y \otimes I_X\alpha$, $L_X(Y \otimes \alpha) \doteq Y \otimes L_X\alpha$ $\forall X, Y \in \mathcal{G}, \forall \alpha \in \mathcal{A}$.

Given a $\mathcal{G}$-operation over a graded algebra $\mathcal{A}$ we define an *algebraic connection* on $\mathcal{A}$ as an element $\omega \in \mathcal{G} \otimes \mathcal{A}^1$ such that $I_X\omega = X \otimes 1 \simeq X$, $L_X\omega = [\omega, X]\ \forall X \in \mathcal{G}$. Given a $\mathcal{G}$-operation we denote its set of algebraic connections by $\mathcal{C}$.

The *curvature* of an algebraic connection is an element $\varrho \in \mathcal{G} \otimes \mathcal{A}^2$ that satisfies $d\omega + (1/2)[\omega, \omega] = \varrho$. In particular this condition implies $d\varrho + [\omega, \varrho] = 0$, $I_X\varrho = 0$, $L_X\varrho = [\varrho, X]\ \forall X \in \mathcal{G}$.

Given $\omega_i \equiv \sum_{\{a_i\}} e_{a_i} \otimes \omega_i^{a_i} \in \mathcal{G} \otimes \mathcal{A}^1$, $i \in N$, we define $\omega_1 \cdots \omega_n \doteq \sum_{\{a_i\}} e_{a_1} \cdots e_{a_n} \otimes \omega_1^{a_1} \cdots \omega_n^{a_n} = \sum_c e_c \otimes (\omega_1 \cdots \omega_n)^c \in \mathcal{G} \otimes \mathcal{A}^n$ with $(\omega_1 \cdots \omega_n)^c \equiv \sum_{\{a_i\}} \gamma^c_{a_1\cdots a_n}\omega_1^{a_1} \cdots \omega_n^{a_n}$.

Now, let us consider bigraded algebras. The definitions will be immediate extensions from the graded case.

A *bigraded commutative algebra* is a pair $(\Upsilon, *)$ such that Υ is an algebra that admits a direct sum decomposition of the type $\Upsilon = \oplus_{(m,n)\in Z\times Z}\Upsilon^{(m,n)}$ and the product $*$ satisfies $\Upsilon^{(m,n)} * \Upsilon^{(r,s)} \subset \Upsilon^{(m+r,n+s)}$, with commutativity meaning $\alpha * \beta = (-1)^{(m+n)(r+s)}\beta * \alpha$ $\forall\, \alpha \in \mathcal{A}^{(m,n)}$, $\forall\, \beta \in \mathcal{A}^{(r,s)}$. Given a bigraded algebra, $\Upsilon^r \equiv \oplus_{m\in Z}\Upsilon^{(m,r-m)}$ defines a graded structure on Υ, i.e. $\Upsilon = \oplus_{r\in Z}\Upsilon^r$.

We also have the same concept of *superderivation* on Υ: a (r, s)-superderivation is a linear map $\Psi : \Upsilon^{(m,n)} \to \Upsilon^{(m+r,n+s)}$ with $\Psi(\alpha\beta) = (\Psi\alpha)\beta + (-1)^{(r+s)(m+n)}\alpha\Psi\beta\ \forall\, \alpha \in \Upsilon^{(m,n)}$. We denote $\mathcal{D}(\Upsilon) \equiv \oplus_{(m,n)\in Z\times Z}\mathcal{D}^{(m,n)}(\Upsilon) = \oplus_{r\in Z}\mathcal{D}^r(\Upsilon)$ where the total degree of a superderivation is given by the sum of its bidegree indices.

A *bigraded commutative differential algebra* is defined as $(\Upsilon, *, \tilde{d})$ with (1) $(\Upsilon, *)$ a bigraded commutative algebra and (2) $\tilde{d}$ a superderivation of total degree 1, $\tilde{d} = \oplus_{m\in Z}\tilde{d}^{(m,1-m)}$.

A *bigraded $\mathcal{G}$-operation* is defined as $(\Upsilon, *, \tilde{d}, \tilde{I}, \tilde{L})$ with (1) $(\Upsilon, *, \tilde{d})$ a bigraded commutative differential algebra and (2) $\tilde{I} : \mathcal{G} \to \mathcal{D}^{-1}(\Upsilon)$ with $\tilde{I}_X \equiv \sum_{m\in Z}\tilde{I}_X^{(m,-1-m)}$, and $\tilde{L} : \mathcal{G} \to \mathcal{D}^0(\Upsilon)$ with $[\tilde{d}, \tilde{I}_X] = \tilde{L}_X \equiv \sum_{m\in Z}\tilde{L}_X^{(m,-m)}$.

An *algebraic connection* on a bigraded $\mathcal{G}$-operation Υ is an element $\tilde{\omega} \in \mathcal{G} \otimes \Upsilon^1$, $\tilde{\omega} \doteq \sum_{k=0}^{D}\tilde{\omega}_k^{1-k}$ satisfying $\tilde{I}_X\tilde{\omega} = X \otimes 1$, $\tilde{L}_X\tilde{\omega} = [\tilde{\omega}, X]$.

The *curvature* of the algebraic connection $\tilde{\omega}$ is an element $\tilde{\varrho} \in \mathcal{G} \otimes \Upsilon^2$, $\tilde{\varrho} \doteq \sum_{i=0}^{D}\tilde{\varrho}_i^{2-i}$ such that $\tilde{d}\tilde{\omega} + (1/2)[\tilde{\omega}, \tilde{\omega}] = \tilde{\varrho}$.

This previous definition of bigraded $\mathcal{G}$-operation is too general. In the next definition we will restrict it in order to fit our purposes.

Definition 1 (BRST $\mathcal{G}$-operation). A BRST $\mathcal{G}$-operation is the structure determined by $(\Upsilon, *, \tilde{d}, \tilde{I}, \tilde{L}, \tilde{\omega}, \tilde{\rho})$ where (1) $(\Upsilon, *, \tilde{d}, \tilde{I}, \tilde{L})$ is a $\mathcal{G}$-operation with

(i) $\Upsilon^{(m,n)} = \{0\}$ if $m < 0$ or $m > D$ with $D \in N$;

(ii) $$\tilde{d} \equiv \sum_{m\in Z^+}\tilde{d}^{(m,1-m)} \doteq b + \underline{d} + \sum_{i=2}^{D}\Delta_i^{1-i}, \qquad \tilde{d}^2 = 0; \tag{142}$$

(iii) $$\tilde{I}_X \equiv \sum_{m\in Z}\tilde{I}_X^{(m,-1-m)} \doteq \tilde{I}_X^{(-1,0)}; \tag{143}$$

(iv) $$\tilde{L}_X \equiv \sum_{m\in Z}\tilde{L}_X^{(m,-m)} \doteq \tilde{L}_X^{(0,0)} \quad \text{with } \tilde{L} = [\tilde{d}, \tilde{I}] \tag{144}$$

and (2) $\tilde{\omega}$ is an algebraic connection on Υ with curvature $\tilde{\rho}$.

Theorem 1. *For a BRST $\mathcal{G}$-operation we have*

$$\tilde{I}_X\underline{d} + \underline{d}\tilde{I}_X = \tilde{L}_X, \tag{145}$$

$$\tilde{I}_X b + b\tilde{I}_X = 0, \tag{146}$$

$$\tilde{I}_X\Delta_i^{1-i} + \Delta_i^{1-i}\tilde{I}_X = 0 \quad \forall\, i \geq 2, \tag{147}$$

$$\tilde{I}_X\tilde{\omega}_i^{1-i} = 0, \quad i \neq 1, \tag{148}$$

$$\tilde{I}_X\tilde{\omega}_1^0 = X \otimes 1, \tag{149}$$

$$\tilde{L}_X\tilde{\omega}_i^{1-i} = [\tilde{\omega}_i^{1-i}, X], \quad 0 \le i \le D, \tag{150}$$

$$\tilde{I}_X\tilde{\varrho}_i^{2-i} = 0, \quad 0 \le i \le D, \tag{151}$$

$$\tilde{L}_X\tilde{\varrho}_i^{2-i} = [\tilde{\varrho}_i^{2-i}, X], \quad 0 \le i \le D. \tag{152}$$

Proof. This follows immediately from Definition 1. □

We extend $\tilde{I}_X, \tilde{L}_X$ to $\mathcal{G}\otimes\Upsilon$ in the same way as we did for the graded case. Note that our definition of BRST $\mathcal{G}$-operation is an extension of that one adopted in [8] in which we allow the differential $\tilde{d}$ to have components Δ_k^{1-k} other than $\tilde{d}^{(0,1)} = b$ and $\tilde{d}^{(1,0)} = \underline{d}$. We also allow the algebraic connection and curvature to contain other component fields in addition to $\tilde{\omega}_0^1, \tilde{\omega}_1^0, \tilde{\varrho}_0^2, \tilde{\varrho}_1^1, \tilde{\varrho}_2^0$.

Finally, we consider $\mathrm{aut}^0(\mathcal{A}) = \{\xi \in \mathcal{G}\otimes\mathcal{A}^0 | L_X\xi = [\xi, X]\,\forall\, X \in \mathcal{G}\}$ that will correspond later on to the concept of the infinitesimal gauge transformations, and $\mathrm{aut}^{*0}(\mathcal{A})$ its dual. In terms of the generators of $\mathcal{G}$ we write $\xi = \sum_a e_a \otimes \xi^a$ with $\xi^a \in \mathcal{A}^0$. Here, the space $\mathcal{A}^0$ is a subalgebra of $\mathcal{A}$, therefore it has a structure of a K-vector space. The space $\mathcal{A}^{*0}$ is then understood as the space of K-linear mappings on $\mathcal{A}^0$. Given $\xi \in \mathrm{aut}^0(\mathcal{A})$ we define

$$I_\xi : \mathcal{A} \to \mathcal{A}, \qquad \alpha \to I_\xi\alpha \doteq \sum_a \xi^a I_a\alpha,$$

$$L_\xi : \mathcal{A} \to \mathcal{A}, \qquad \alpha \to L_\xi\alpha \doteq \sum_a ((d\xi^a) I_a\alpha + \xi^a L_a\alpha)$$

and we extend them to $\mathcal{G}\otimes\mathcal{A}$ as $I_\xi(X\otimes\alpha) = X\otimes I_\xi\alpha$, $L_\xi(X\otimes\alpha) = X\otimes L_\xi\alpha$. In particular, they act on the space of algebraic connections $\mathcal{C} \subset \mathcal{G}\otimes\mathcal{A}^1$ giving

$$L_\xi(\omega) = d\xi + [\omega, \xi], \tag{153}$$

$$I_\xi(\omega) = \xi. \tag{154}$$

It is immediate to check that

$$I_X L_\xi\omega = 0, \tag{155}$$

$$L_X L_\xi\omega = [L_\xi\omega, X], \tag{156}$$

therefore $L_\xi\omega$ is not an algebraic connection. We obtain an algebraic connection through the combination $\omega + L_\xi\omega$. Here $L_\xi\omega$ is interpreted as the infinitesimal gauge transformation of ω. Given an algebraic connection ω we also define

$$D_\omega : \mathcal{G}\otimes\mathcal{A} \to \mathcal{G}\otimes\mathcal{A}, \qquad D_\omega = d + [\omega, \dots] \tag{157}$$

and we have $L_\xi\omega = D_\omega\xi$. It is straightforward to derive the following properties:

$$D_\omega L_\xi\omega = [\rho, \xi], \tag{158}$$

$$I_X D_\omega L_\xi\omega = 0, \qquad L_X D_\omega L_\xi\omega = [D_\omega L_\xi\omega, X]. \tag{159}$$

5.2. An example of bigraded algebra: $(\mathcal{K}, *)$

Let us denote $\mathcal{K}^{(0,0)} \equiv \{\mathbf{1} : \mathcal{C} \to \mathcal{A}^0, \mathbf{1}(\omega) = 1 \in \mathcal{A}^0\, \forall\, \omega \in \mathcal{C}\}$:

$$\mathcal{K}^{(m,n)} = \begin{cases} \mathcal{F}(\mathcal{C} \times (\mathrm{aut}^0(\mathcal{A}))^n, \mathcal{A}^m) \simeq \mathcal{F}(\mathcal{C}, \bigwedge^n(\mathrm{aut}^{0*}(\mathcal{A})) \otimes \mathcal{A}^m) & \text{if } n \geq 0, \\ \mathcal{F}(\mathcal{C} \times (\mathrm{aut}^{0*}(\mathcal{A}))^n, \mathcal{A}^m) \simeq \mathcal{F}(\mathcal{C}, \bigwedge^n(\mathrm{aut}^{0}(\mathcal{A})) \otimes \mathcal{A}^m) & \text{if } n < 0 \end{cases} \tag{160}$$

and $\mathcal{K} \doteq \oplus_{(m,n)\in Z^+\times Z}\, \mathcal{K}^{(m,n)}$. Here, $\mathcal{F}(\mathcal{C} \times (\mathrm{aut}^0(\mathcal{A}))^n, \mathcal{A}^m)$ denotes the space of n-linear antisymmetric maps in $\mathrm{aut}^0(\mathcal{A})$ with values in $\mathcal{A}^m$, and analogously $\mathcal{F}(\mathcal{C}\times(\mathrm{aut}^{0*}(\mathcal{A}))^n, \mathcal{A}^m)$ denotes the space of n-linear antisymmetric maps in $\mathrm{aut}^{0*}(\mathcal{A})$ with values in $\mathcal{A}^m$.

We write $\tau^n_m \equiv \sum_{\{\hat{\tau},\omega\}} \hat{\tau}^n \otimes w_m$ with $\hat{\tau}^n : \mathcal{C} \to \bigwedge^n(\mathrm{aut}^{0*}(\mathcal{A}))$ if $n > 0$ or $\hat{\tau}^n : \mathcal{C} \to \bigwedge^n(\mathrm{aut}^{0}(\mathcal{A}))$ if $n < 0$ and $w_m : \mathcal{C} \to \mathcal{A}^m$. The last sum is done over decomposable elements $\{\hat{\tau}^n, w_m\}$. Let us introduce a product among elements of $\mathcal{F}(\mathcal{C}, \bigwedge(\mathrm{aut}^{0*}(\mathcal{A}))) \cup \mathcal{F}(\mathcal{C}, \bigwedge(\mathrm{aut}^{0}(\mathcal{A})))$,

Definition 2. Let $n, n' \in N$. Given $\hat{\tau}^n, \hat{\tau}^{n'}, \hat{\tau}^{-n}, \hat{\tau}^{-n'} \in \mathcal{F}(\mathcal{C}, \bigwedge(\mathrm{aut}^{0*}(\mathcal{A}))) \cup \mathcal{F}(\mathcal{C}, \bigwedge(\mathrm{aut}^{0}(\mathcal{A})))$ we define

$$\hat{\tau}^n \bigstar \hat{\tau}^{n'}(\omega; \xi_1, \dots, \xi_{n+n'}) \doteq \frac{1}{(n+n')!} \sum_{\sigma\in P_{n+n'}} \epsilon_\sigma \hat{\tau}^n(\omega; \xi_{\sigma_1}, \dots, \xi_{\sigma_n}) \hat{\tau}^{n'}(\omega; \xi_{\sigma_{n+1}}, \dots, \xi_{\sigma_{n+n'}}), \tag{161}$$

$$\hat{\tau}^{-n} \bigstar \hat{\tau}^{-n'}(\omega; \xi^*_1, \dots, \xi^*_{n+n'}) \doteq \frac{1}{(n+n')!} \sum_{\sigma\in P_{n+n'}} \epsilon_\sigma \hat{\tau}^{-n}(\omega; \xi^*_{\sigma_1}, \dots, \xi^*_{\sigma_n}) \hat{\tau}^{-n'}(\omega; \xi^*_{\sigma_{n+1}}, \dots, \xi^*_{\sigma_{n+n'}}), \tag{162}$$

$$\left.\begin{aligned} &\hat{\tau}^{-n'} \bigstar \hat{\tau}^{n}(\omega; \xi^*_1, \dots, \xi^*_{n'-n}) \doteq \hat{\tau}^{-n'}(\omega; \xi^*_1, \dots, \xi^*_{n'-n}, \hat{\tau}^n(\omega)) \\ &\hat{\tau}^{n} \bigstar \hat{\tau}^{-n'} \doteq (-1)^{nn'} \hat{\tau}^{-n'} \bigstar \hat{\tau}^{n} \end{aligned}\right\} \quad \text{for } n' > n, \tag{163}$$

$$\left.\begin{aligned} &\hat{\tau}^{n} \bigstar \hat{\tau}^{-n'}(\omega; \xi_1, \dots, \xi_{n-n'}) \doteq \hat{\tau}^{n}(\omega; \xi_1, \dots, \xi_{n-n'}, \hat{\tau}^{-n'}(\omega)) \\ &\hat{\tau}^{-n'} \bigstar \hat{\tau}^{n} \doteq (-1)^{nn'} \hat{\tau}^{n} \bigstar \hat{\tau}^{-n'} \end{aligned}\right\} \quad \text{for } n > n'. \tag{164}$$

Notice that fixing $(n'-n)$ elements $\xi^*_1, \dots, \xi^*_{n'-n}$ on the right-hand side of (163) we have $\hat{\tau}^{-n'}$ as a n-linear antisymmetric map on $(\mathrm{aut}^{*0}(\mathcal{A}))$. For simplicity let us consider $\hat{\tau}^n(\omega)$ as a decomposable element $\theta^*_1 \wedge \cdots \wedge \theta^*_n$. Using the isomorphism $\mathcal{F}_{\mathrm{linear}}(\bigwedge^n(\mathrm{aut}^{*0}(\mathcal{A})), K) \simeq \mathcal{F}(\mathrm{aut}^{*0}(\mathcal{A}) \times \cdots \times \mathrm{aut}^{*0}(\mathcal{A}), K)$ (the rhs denoting the space of n-linear antisymmetric maps in $\mathrm{aut}^{0*}(\mathcal{A})$) we interpret $\hat{\tau}^{-n'}(\omega; \xi^*_1, \dots, \xi^*_{n'-n}, \hat{\tau}^n(\omega)) = \hat{\tau}^{-n'}(\omega; \xi^*_1, \dots, \xi^*_{n'-n}, \theta^*_1, \dots, \theta^*_n)$ that is the exact meaning to the rhs of (163).

Definition 3. We define a product in $\mathcal{K}$ as

$$* : \mathcal{K}^{(m,n)} \times \mathcal{K}^{(m',n')} \to \mathcal{K}^{(m+m',n+n')}$$

$$(\tau^n_m, \tau^{n'}_{m'}) \to \tau^n_m * \tau^{n'}_{m'} \doteq \hat{\tau}^n \star \hat{\tau}^{n'} \otimes (-1)^{mn'} w_m \wedge w_{m'}. \tag{165}$$

Theorem 2. $(\mathcal{K}, *)$ *is a bigraded commutative algebra*.

Proof. The product $*$ satisfies $\mathcal{K}^{(m,n)} * \mathcal{K}^{(m',n')} \subset \mathcal{K}^{(m+m',n+n')}$ which makes $\mathcal{K} \doteq \oplus_{(m,n)\in Z^+\times Z} \mathcal{K}^{(m,n)}$ a graded algebra. The product $\star$ satisfies $\hat{\tau}^n \star \hat{\tau}^{n'} = (-1)^{nn'} \hat{\tau}^{n'} \star \hat{\tau}^n$, and we have $\tau^n_m * \tau^{n'}_{m'} = (-1)^{(n+m)(n'+m')} \tau^{n'}_{m'} * \tau^n_m$, i.e. $*$ is commutative. □

5.2.1. *Extending $(\mathcal{K}, *)$ to a bigraded $\mathcal{G}$-operation*

Let $(\mathcal{A}, \cdot, d, I, L)$ be a Z^+-graded $\mathcal{G}$-operation. Define on $\mathcal{K}$ the maps $\underline{d}$, $\tilde{I}_X$, $\tilde{L}_X$ $\forall X \in \mathcal{G}$ as

$$\underline{d} : \mathcal{K}^{(m,n)} \to \mathcal{K}^{(m+1,n)}$$

$$(\underline{d}\alpha^n_m)(\omega; \zeta_1, \dots, \zeta_n) \doteq d(\alpha^n_m(\omega; \zeta_1, \dots, \zeta_n)), \tag{166}$$

$$\tilde{I}_X : \mathcal{K}^{(m,n)} \to \mathcal{K}^{(m-1,n)}$$

$$(\tilde{I}_X\alpha^n_m)(\omega; \zeta_1, \dots, \zeta_n) \doteq I_X(\alpha^n_m(\omega; \zeta_1, \dots, \zeta_n)), \tag{167}$$

$$\tilde{L}_X : \mathcal{K}^{(m,n)} \to \mathcal{K}^{(m,n)}$$

$$(\tilde{L}_X\alpha^n_m)(\omega; \zeta_1, \dots, \zeta_n)) \doteq L_X(\alpha^n_m(\omega; \zeta_1, \dots, \zeta_n)) \tag{168}$$

$\forall\, \alpha^n_m \in \mathcal{K}^{(m,n)}$ and with ζ_i, $i = 1, \dots, n$ denoting elements of either $\mathrm{aut}^0(\mathcal{A})$ or $\mathrm{aut}^{0*}(\mathcal{A})$. $\tilde{I}$ and $\tilde{L}$ satisfy

$$\tilde{I}_{[X,Y]} = \tilde{L}_X\tilde{I}_Y - \tilde{I}_Y\tilde{L}_X, \tag{169}$$

$$\tilde{L}_{[X,Y]} = \tilde{L}_X\tilde{L}_Y - \tilde{L}_Y\tilde{L}_X \quad \forall\, X, Y \in \mathcal{G} \tag{170}$$

and this makes $(\mathcal{K}, *, \underline{d}, \tilde{I}, \tilde{L})$ a bigraded $\mathcal{G}$-operation.

5.3. *A particular example of a BRST $\mathcal{G}$-operation: $\mathcal{H}$*

Let us define the following elements of $\mathcal{G} \otimes \mathcal{K}$:

- $\tilde{\varphi}^1_0 \doteq \tilde{c} \equiv \sum_a e_a \otimes \tilde{c}^a \in \mathcal{G} \otimes \mathcal{F}(\mathcal{C} \times \mathrm{aut}^0(\mathcal{A}), \mathcal{A}^0)$:

$$\tilde{c}^a(\omega; \xi) \doteq \xi^a + \sum_{i=1}^{N} \theta^{*i}(\xi)(I_{\theta_i}\omega)^a = \xi^a + \sum_{i=1}^{N} \theta^{*i}(\xi)\theta^a_i; \tag{171}$$

- $\tilde{\varphi}^0_1 \doteq \tilde{A} \equiv \sum_a e_a \otimes \tilde{A}^a \in \mathcal{G} \otimes \mathcal{F}(\mathcal{C}, \mathcal{A}^1)$:

$$\tilde{A}^a(\omega) \doteq \omega^a + \sum_{i=1}^{N} A^i(L_{\theta_i}\omega)^a, \quad A^i \in K; \tag{172}$$

- $\tilde{\varphi}_k^{1-k} \equiv \sum_a e_a \otimes \tilde{\varphi}_k^{a,1-k} \in \mathcal{G} \otimes \mathcal{F}(\mathcal{C} \times (\mathrm{aut}^{0*}(\mathcal{A}))^{k-1}; \mathcal{A}^k)$ $k \geq 2$:

$$\tilde{\varphi}_k^{a,1-k}(\omega, \xi_1^*, \dots, \xi_{k-1}^*) = \sum_{\{i_1,\dots,i_{k-1}\}\subset\{1,\dots,N\}} \theta_{i_1} \wedge \dots \wedge \theta_{i_{k-1}}(\xi_1^* \dots \xi_{k-1}^*) \otimes (D_\omega(L_{\theta_{i_1}}\omega \cdots L_{\theta_{i_{k-1}}}\omega))^a; \tag{173}$$

- $\tilde{\eta}_0^2 \doteq \tilde{\phi} \equiv \sum_a e_a \otimes \tilde{\phi}^a \in \mathcal{G} \otimes \mathcal{F}(\mathcal{C} \times (\mathrm{aut}^0(\mathcal{A}))^2, \mathcal{A}^0)$:

$$\tilde{\phi}^a(\omega, \xi_1, \xi_2) = \sum_{\{i_1,i_2\}\subset\{1,\dots,N\}} \theta^{*i_1} \wedge \theta^{*i_2}(\xi_1, \xi_2) \otimes [\theta_{i_1}, \theta_{i_2}]^a; \tag{174}$$

- $\tilde{\eta}_1^1 \doteq \tilde{\psi} \equiv \sum_a e_a \otimes \tilde{\psi}^a \in \mathcal{G} \otimes \mathcal{F}(\mathcal{C} \times (\mathrm{aut}^0(\mathcal{A}))^1, \mathcal{A}^1)$:

$$\tilde{\psi}^a(\omega; \xi) = \sum_{i=1}^N \theta^{*i}(\xi) \otimes (L_{\theta_i}\omega)^a; \tag{175}$$

- $\tilde{\eta}_2^0 \doteq \tilde{B} \equiv \sum_a e_a \otimes \tilde{B}^a \in \mathcal{G} \otimes \mathcal{F}(\mathcal{C}, \mathcal{A}^2)$:

$$\tilde{B}^a(\omega) = \tilde{F}^a(\omega) + \sum_{i,j=1}^N B^{ij}(L_{\theta_i}\omega L_{\theta_j}\omega)^a \quad \text{with } \tilde{F} = \underline{d} + \frac{1}{2}[\tilde{A}, \tilde{A}], \quad B^{ij} \in K; \tag{176}$$

- $\tilde{\eta}_k^{2-k} \equiv \sum_a e_a \otimes \tilde{\eta}_k^{a,2-k} \in \mathcal{G} \otimes \mathcal{F}(\mathcal{C} \times (\mathrm{aut}^{0*}(\mathcal{A}))^{k-2}, \mathcal{A}^k)$, $k \geq 3$:

$$\tilde{\eta}_k^{a,2-k}(\omega; \xi_1^*, \dots, \xi_{k-2}^*) = \sum_{\{i_1,\dots,i_{k-2}\}\subset\{1,\dots,N\}} \theta_{i_1} \wedge \dots \wedge \theta_{i_{k-2}}(\xi_1^*, \dots, \xi_{k-2}^*) \otimes (L_{\theta_{i_1}}\omega \cdots L_{\theta_{i_{k-2}}}\omega\rho)^a \tag{177}$$

$\forall\, \xi_k \in \mathrm{aut}^0(\mathcal{A})$, $\forall\, \xi_k^* \in \mathrm{aut}^{0*}(\mathcal{A})$ and for $\theta_i \in \mathrm{aut}^0(\mathcal{A})$ and $\theta^{*i} \in \mathrm{aut}^{0*}(\mathcal{A})$, $i = 1, \dots, N$. The integer N may denote any number of elements of $\mathrm{aut}^0(\mathcal{A})$ and its dual. In this sense, to any choice of N pairs (θ_i, θ^{*i}) we have a specific form for $\tilde{\varphi}_i^{1-i}$, $\tilde{\eta}_i^{2-i}$ given by (171)–(177). In addition, given a certain field $\tilde{\varphi}_i^{1-i}$ or $\tilde{\eta}_i^{2-i}$ we have associated a finite sequence of fields

$$\begin{array}{c} \tilde{c} \to \tilde{A} \to \cdots \tilde{\varphi}_k^{1-k} \to \cdots \tilde{\varphi}_N^{1-N} \\ \downarrow\uparrow \\ \tilde{\phi} \to \tilde{\psi} \to \cdots \tilde{\eta}_k^{2-k} \to \cdots \tilde{\eta}_N^{2-N}, \end{array} \tag{178}$$

each of them defined by (171)–(177) in terms of the same N pairs (θ_i, θ^{*i}) that appear in $\tilde{\varphi}_i^{1-i}$ or $\tilde{\eta}_i^{2-i}$.

From (153)–(156) we notice that they also satisfy

$$\tilde{I}_X \tilde{\varphi}_i^{1-i} = 0, \quad i \neq 1, \tag{179}$$

$$\tilde{I}_X\tilde{\varphi}_1^0 = X\otimes 1, \tag{180}$$

$$\tilde{L}_X\tilde{\varphi}_i^{1-i} = [\tilde{\varphi}_i^{1-i}, X], \quad 0\le i\le D, \tag{181}$$

$$\tilde{I}_X\tilde{\eta}_i^{2-i} = 0, \quad 0\le i\le D, \tag{182}$$

$$\tilde{L}_X\tilde{\eta}_i^{2-i} = [\tilde{\eta}_i^{2-i}, X], \quad 0\le i\le D. \tag{183}$$

Once again, let $(\mathcal{A}, ., d, I, L)$ be a $\mathcal{G}$-operation and $\mathcal{C}\subset\mathcal{G}\otimes\mathcal{A}^1$ be the space of algebraic connections on $\mathcal{A}$. We introduce a particular BRST $\mathcal{G}$-operation as follows.

Definition 4. $(\mathcal{H}, *, \tilde{d}, \tilde{I}, \tilde{L}, \tilde{\omega}, \tilde{\rho})$ is a BRST $\mathcal{G}$-operation with (1) $(\mathcal{H}, *, \tilde{d}, \tilde{I}, \tilde{L})$ a $\mathcal{G}$-operation such that

(i) $\mathcal{H}$ is the subalgebra of $\mathcal{K}$ generated by $\{\tilde{\varphi}_i^{1-i}, \underline{d}\tilde{\varphi}_i^{1-i}, \tilde{\eta}_i^{2-i}, \underline{d}\tilde{\eta}_i^{2-i}\}_{i=1,\dots,N}$. The graded structure of $\mathcal{H}$ is obtained from the graded structure of $\mathcal{K}$ and we write $\mathcal{H} = \oplus_{(m,n)\in Z^+\times Z}\mathcal{H}^{(m,n)}$ with $\mathcal{H}^{(m,n)} = \mathcal{K}^{(m,n)}\cap\mathcal{H}$.
(ii) The product in $\mathcal{H}$ is defined by the same product in $\mathcal{K}$ as given in (165).
(iii) The differential in $\mathcal{H}$ is a map $\tilde{d}:\mathcal{H}^{(m,n)}\to\mathcal{H}^{m+n+1}\doteq\oplus_{r\in Z^+}\mathcal{H}^{(r,m+n+1-r)}$, $\tilde{d}\equiv\sum_{i=0}^{D}\Delta_i^{1-i}\doteq b+\underline{d}+\sum_{i=2}^{D}\Delta_i^{1-i}$ with $\tilde{d}^2 = 0$ and $\underline{d}$ a superderivation of degree (1,0) defined as (166). The BRST operator is a superderivation of bidegree (0,1), $b:\mathcal{H}^{(m,n)}\to\mathcal{H}^{(m,n+1)}$ defined by (44)–(47), and $\Delta_i^{1-i}:\mathcal{H}^{(m,n)}\to\mathcal{H}^{(m+i,n-i+1)}$ is a superderivation of degree (i,1-i) defined as in (31) with δ given as in (48)–(55).[4]
(iv) The interior product $\tilde{I}$ is given by (167) and the Lie derivative $\tilde{L}$ is given by (168). (2) The algebraic connection and curvature are defined as $\tilde{\omega} = \sum_{i=0}^{N}\tilde{\varphi}_i^{1-i}$ and $\tilde{\varrho} = \sum_{i=0}^{N}\tilde{\eta}_i^{2-i}$. From (179)–(183) we obtain that $\tilde{I}_X\tilde{\omega} = X\otimes 1$, $\tilde{L}_X\tilde{\omega} = [\tilde{\omega}, X]$, $\tilde{I}_X\tilde{\varrho} = 0$, $\tilde{L}_X\tilde{\varrho} = [\tilde{\varrho}, X]$.

The zero curvature limit is a particular case of the previous construction when $\mathcal{H}$ is generated by $\{\tilde{\varphi}_i^{1-i}, \underline{d}\tilde{\varphi}_i^{1-i}\}_{i=1,\dots,N}$ and the algebraic connection satisfies $\tilde{d}\tilde{\omega}+(1/2)[\tilde{\omega},\tilde{\omega}] = 0$.

6. The gauge group and the gauge algebra

In this section we review the concepts of gauge group and gauge algebra. Our main purpose is to set up our notations and give an intuitive development of these concepts.

Let $\pi: P\to\mathcal{M}$ be a principal fiber bundle with structure group G. Let us denote $\mathcal{G}$ the Lie algebra of G and $\tilde{R} = P\times G\to P$, $\tilde{R}_g: P\to P$ the right action of G on P. For $X\in\mathcal{G}$ we have associated a $\tilde{X}\in\mathcal{F}_{\text{fund}}^{(1,0)}(P)$, with $\mathcal{F}_{\text{fund}}^{(1,0)}(P)$ the space of fundamental vector fields on P. Given $f\in\mathcal{F}(P,R)$, $\tilde{X}\in\mathcal{F}_{\text{fund}}^{(1,0)}(P)$ we define $(f\cdot\tilde{X})(p) = f(p)\tilde{X}(p)$. This turns $\mathcal{F}_{\text{fund}}^{(1,0)}(P)$ into a $\mathcal{F}(P,R)$-module that we denote as $\aleph_{\text{fund}}(P)$. We have the isomorphisms $\mathcal{F}(P,\mathcal{G})\simeq\mathcal{F}(P,R)\otimes\mathcal{G}\simeq\aleph_{\text{fund}}(P)$ where the second isomorphism is defined as $\mathcal{F}(P,R)\otimes\mathcal{G}\ni f\otimes X\leftrightarrow f\cdot\tilde{X}\in\aleph_{\text{fund}}(P)$.

[4] Here $\underline{d}$ replaces d in the expressions for b, Δ, δ given in (44)–(55).

The *gauge group* of P is denoted by $\underline{G}$ and can be identified in three equivalent ways: $\underline{G} = \mathrm{Aut}_v(P) \simeq \mathcal{F}_{\mathrm{eq}}(P, G) \simeq \Gamma(AdP)$ [23,26]. Here, $f \in \mathrm{Aut}_v(P) \subset \mathrm{Diff}(P)$ is such that $\pi \circ f = \pi$, $\tilde{R}_g \circ f = f \circ \tilde{R}_g \;\forall\, g \in G$. The group structure of $\mathrm{Aut}_v(P)$ is defined by the composition of maps. Next, $\tilde{f} \in \mathcal{F}_{\mathrm{eq}}(P, G)$ is a map $\tilde{f} : P \to G$ such that $\tilde{f}(pg) = Ad(g^{-1})\tilde{f}(p)$. The group structure of $\mathcal{F}_{\mathrm{eq}}(P, G)$ is given by pointwise multiplication, $(\tilde{f} \cdot \tilde{f}')(p) = \tilde{f}(p)\tilde{f}'(p)$. Finally, $\Gamma(AdP)$ denotes the space of C^∞ sections on the adjoint bundle $AdP \equiv P \times_{Ad} G$ with Ad the adjoint map on G [23,26]. In this work we will consider just the first two identifications.

The $1-1$ map between $\mathrm{Aut}_v(P)$ and $\mathcal{F}_{\mathrm{eq}}(P, G)$ is defined as follows. Given $f \in \mathrm{Aut}_v(P)$ we can define $\tilde{f} \in \mathcal{F}_{\mathrm{eq}}(P, G)$ [8,23,26] such that $f(p) = p\tilde{f}(p) \;\forall\, p \in P$. Conversely, given $\tilde{f} \in \mathcal{F}_{\mathrm{eq}}(P, G)$ we define $f \in \mathrm{Aut}_v(P)$, $f = \tilde{R} \circ (\mathrm{id}, \tilde{f})$ with *id* the identity map on P. Those two maps allow us to identify $\mathrm{Aut}_v(P) \simeq \mathcal{F}_{\mathrm{eq}}(P, G)$.

The concept of tangent space on a space of maps [27] can be used to define the tangent space of $\mathrm{Aut}_v(P)$ at f. This will give a definition for the gauge algebra in the same way as one defines the Lie algebra of a Lie group as the tangent space to the identity. We define $X_f \in T_f(\mathrm{Aut}_v(P))$ as a map $X_f : P \to T_f(P)$ such that $X_f(p) \in T_{f(p)}(P)$ with

$$X_f \equiv \left.\frac{\mathrm{d}}{\mathrm{d}t}\phi_t\right|_{t=0} \tag{184}$$

and

- $\phi_t \in \mathrm{Aut}_v(P)$ (i.e. $\pi \circ \phi_t = \pi$, $\tilde{R}_g \circ \phi_t = \phi_t \circ \tilde{R}_g$, $\phi_0 \equiv f$);
- $\phi_p : R \to P$ is a differentiable curve in P such that $\phi_p(t) = \phi_t(p)$.

Then we have

$$\pi_* X_f(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\pi \circ \phi_t(p)\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\pi(p)\right|_{t=0} = 0,$$

i.e. $X_f(p) \in V_{f(p)} \equiv T_{f(p)}(\pi^{-1}(x))$, $(\pi(p) = \pi(f(p)) = x)$. Also

$$\tilde{R}_{g*}X_f(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\tilde{R}_g \circ \phi_t(p)\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(p \cdot g)\right|_{t=0} = X_f(p \cdot g),$$

i.e. $\tilde{R}_{g*}X_f = X_f$.

Now, since f is a 1–1 map we note that $p \neq p' \Rightarrow X_f(p) \in T_{f(p)} \neq T_{f(p')} \ni X_f(p')$, therefore it is possible to choose a vector field $\tilde{X} \in \mathcal{F}^{(1,0)}(P)$ such that $\forall\, p \in P$, $X_f(p) \equiv \tilde{\varepsilon}_{f(p)}\tilde{X}(f(p))$ ($\tilde{\varepsilon}_{f(p)} \in R$), or $X_f \equiv (\tilde{\varepsilon} \cdot \tilde{X}) \circ f$ with $\tilde{\varepsilon} \in \mathcal{F}(P, R)$, i.e. $X_f \in \aleph(P)$. The first condition restricts $\tilde{X} \in \mathcal{F}^{(1,0)}_{\mathrm{fund}}(P)$ and consequently $X_f = (\tilde{\varepsilon} \cdot \tilde{X}) \circ f \in \aleph_{\mathrm{fund}}(P)$. The second condition gives $\tilde{\varepsilon}(f(p))\tilde{R}_{g*}\tilde{X}_{f(p)} = \tilde{\varepsilon}(f(pg))\tilde{X}_{f(pg)}$. Let $\{\tilde{e}, i = 1, \ldots, \dim G\}$ be a basis for $\mathcal{F}^{(1,0)}_{\mathrm{fund}}(P)$. Then $\tilde{X} = \lambda^i\tilde{e}_i$ and $X_f \equiv (\tilde{\varepsilon} \cdot \tilde{X}) \circ f = (\tilde{\varepsilon}^i \cdot \tilde{e}_i) \circ f$ with $\tilde{\varepsilon}^i = \lambda^i\tilde{\varepsilon}$. We have then characterized.

- $T_f(\mathrm{Aut}_v(P)) = \{(\tilde{\varepsilon}^i \cdot \tilde{e}_i) \circ f | \tilde{\varepsilon}^i \in \mathcal{F}(P, R),\;\; \tilde{e}_i \in \mathcal{F}^{(1,0)}_{\mathrm{fund}}(P)\}$ with

$$\tilde{\varepsilon}^i(f(p))\tilde{R}_{g*}\tilde{e}_i(f(p)) = \tilde{\varepsilon}^i(f(pg))\tilde{e}_i(f(pg)). \tag{185}$$

Let us now consider the tangent space on the identity map $I \in \mathrm{Aut}_v(P)$. From the previous development we obtain that $X_I \in T_I(\mathrm{Aut}_v(P))$ has the form $X_I = \tilde{\varepsilon}^i \tilde{e}_i$ and should satisfy $\tilde{\varepsilon}^i(p)\tilde{R}_{g*}\tilde{e}_i(p) = \tilde{\varepsilon}^i(pg)\tilde{e}_i(pg)$. We then have

$$\tilde{\varepsilon}^i(p)\tilde{R}_{g*}\tilde{e}_i(p) = \tilde{\varepsilon}^i(p)\tilde{R}_{g*}\tilde{R}_{p*}e_i(e) = \tilde{\varepsilon}^i(p)\tilde{R}_{p*}R_{g*}e_i(e) = \tilde{R}_{p*}(\tilde{\varepsilon}^i(p)R_{g*}e_i(e)), \tag{186}$$

$$\tilde{\varepsilon}^i(pg)\tilde{e}_i(pg) = \tilde{\varepsilon}^i(pg)\tilde{R}_{pg*}e_i(e) = \tilde{\varepsilon}^i(pg)\tilde{R}_{p*}L_{g*}e_i(e) = \tilde{R}_{p*}(\tilde{\varepsilon}^i(pg)L_{g*}e_i(e)). \tag{187}$$

Since the action of G on P is free we obtain, $\tilde{\varepsilon}^i(p)R_{g*}e_i(e) = \tilde{\varepsilon}^i(pg)L_{g*}e_i(e)$ and then $ad(g^{-1})(\tilde{\varepsilon}^i(p)e_i(e)) = \tilde{\varepsilon}^i(pg)e_i(e)$. We then define $\mathcal{F}_{\mathrm{eq}}(P, \mathcal{G})$ as the set of elements of this type, i.e. $\mathcal{F}_{\mathrm{eq}}(P, \mathcal{G}) = \{\tilde{\varepsilon} = \tilde{\varepsilon}^i \otimes e_i | \tilde{\varepsilon}(pg) = ad(g^{-1})\tilde{\varepsilon}(p),\ \ \tilde{\varepsilon}^i \in \mathcal{F}(P, R),\ \ e_i \in \mathcal{G}\}$. This result defines an isomorphism $T_I(\mathrm{Aut}_v(P)) \simeq \mathcal{F}_{\mathrm{eq}}(P, \mathcal{G})$ that provides another description for the gauge algebra $\underline{\mathcal{G}}$.

Here, for the case of $T_I(\mathrm{Aut}_v(P))$ let us find an explicit form for the diffeomorphisms ϕ_t (184). Consider $X_I = \tilde{\varepsilon}^i \tilde{e}_i \equiv (\mathrm{d}/\mathrm{d}t)\phi_t|_{t=0}$. Let us take local charts (U_α, ψ_α) of G and (V_β, χ_β) of P in terms of which we can write $\tilde{R}^r_p(x) \equiv \chi^r \circ \tilde{R}_p \circ \psi^{-1}(x)$. We denote $\psi(g) = x \equiv (x^1, \dots, x^n)$, $\psi^i(g) = x^i$ and $\chi(p) = y \equiv (y^1, \dots, y^m)$, $\chi^r(p) = y^r$. We can write $\tilde{e}_i(p) \equiv \tilde{R}_{p*}e_i(e) = (\partial \tilde{R}^r_p(x)/\partial x^i)|_{\psi(e)}(\partial/\partial y^r)|_{\chi(p)}$ and $\tilde{\varepsilon}^i(p) = (\mathrm{d}/\mathrm{d}t)\psi^i \circ \exp(t\tilde{\varepsilon}(p))|_{t=0}$ then

$$\begin{aligned} X_I(p) = \tilde{\varepsilon}^i(p)\tilde{e}_i(p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\psi^i \circ \exp(t\tilde{\varepsilon}(p))\right|_{t=0} \left.\frac{\partial \tilde{R}^r_p(x)}{\partial x^i}\right|_{\psi(e)} \left.\frac{\partial}{\partial y^r}\right|_{\chi(p)} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\tilde{R}^r_p(\exp(t\tilde{\varepsilon}(p)))\right|_{t=0} \left.\frac{\partial}{\partial y^r}\right|_{\chi(p)} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\tilde{R}_p(\exp(t\tilde{\varepsilon}(p)))\right|_{t=0} \end{aligned} \tag{188}$$

that suggest us to define $\phi_t = \tilde{R}_{\exp(t\tilde{\varepsilon})}$ with $\phi_t(p) \doteq \tilde{R}(p, (\exp(t\tilde{\varepsilon}(p)))) = \tilde{R}_p(\exp(t\tilde{\varepsilon}(p)))$. (188) agrees with the same expression given in Schmid [26] for the elements $Z_{\tilde{\varepsilon}}$ of the gauge algebra.

7. An explicit realization for $\mathcal{H}$

Let $P(\mathcal{M}, G)$ be a principal fiber bundle with structure group G. We define $\mathcal{A} \doteq \Omega(P) = \oplus_{r \in Z^+} \Omega^r(P) \equiv \oplus_{r \in Z^+} \mathcal{A}^r$. Considering the interior product and Lie derivative on $\Omega(P)$ we define $\forall\, X$ (with $\mathcal{G} \ni X \leftrightarrow \tilde{X} \in \mathcal{F}^{(1,0)}_{\mathrm{fund}}(P)$):

$$I_X \doteq I_{\tilde{X}}, \qquad L_X \doteq L_{\tilde{X}}$$

that satisfies conditions $I_{[X,Y]} = L_X I_Y - I_Y L_X$, $L_{[X,Y]} = L_X L_Y - L_Y L_X$ $\forall\, X, Y \in \mathcal{G}$. Therefore, taking the multiplication on $\Omega(P)$ as the exterior product and the differential as the exterior derivative it is straightforward to see that $(\Omega(P), \wedge, d, I, L)$ becomes a $\mathcal{G}$-operation.

A connection on P is an element $\omega \in \mathcal{G} \otimes \Omega^1(P)$ that satisfies $\tilde{R}_g^* \omega = ad(g^{-1}) \cdot \omega$, $\omega(\tilde{X}) = X$ with $ad(g) = L_{g*} R_{g^{-1}*}$. These conditions imply

$$L_X \omega = [\omega, X], \qquad I_X \omega = X \otimes 1.$$

With the choice $\mathcal{A}^r \doteq \Omega^r(P)$ we have that $\mathrm{aut}^0(\mathcal{A})$ is the gauge algebra, i.e. $\mathrm{aut}^0(\mathcal{A}) \equiv \mathcal{F}_{\mathrm{eq}}(P, \mathcal{G})$. Indeed, let $\xi \in \mathcal{F}_{\mathrm{eq}}(P, \mathcal{G})$. Since $\mathcal{F}(P, \mathcal{G}) \simeq \mathcal{G} \otimes \Omega^0(P)$ we can write $\xi = \sum_a e_a \otimes \xi^a$. Then $\forall X \in \mathcal{G}$, $L_X \xi = [\xi, X]$ (see [5]). We have analogue expressions for $L_\xi : \Omega(P) \to \Omega(P)$ (153) and $D_\omega : \mathcal{G} \otimes \Omega(P) \to \mathcal{G} \otimes \Omega(P)$ (157).

The components of the algebraic connection and the curvature will depend on the assignment of at least $N = D$ linearly independent elements of $\mathrm{aut}^0(P)$ and its dual $\mathrm{aut}^{*0}(P)$. Their definition follow the same procedure given in (171)–(177) and they are functions ($0 \le i \le q \le D$, with $q \in Z$ and D the spacetime dimension):

$$\begin{aligned}
&\tilde{c}^a \doteq \tilde{\varphi}_0^{a,1} \in \mathcal{H}^{(0,1)} \subset \mathcal{F}(\mathcal{C} \times \underline{\mathcal{G}}, \Omega^0(P)), && \tilde{A}^a \doteq \tilde{\varphi}_1^{a,0} \in \mathcal{H}^{(1,0)} \subset \mathcal{F}(\mathcal{C}, \Omega^1(P)),\\
&\tilde{\varphi}_i^{a,1-i} \in \mathcal{H}^{(i,1-i)} \subset \mathcal{F}(\mathcal{C} \times \underline{\mathcal{G}}^{*i-1}, \Omega^i(P)), && \tilde{\phi}^a \doteq \tilde{\eta}_0^{a,2} \in \mathcal{H}^{(0,2)} \subset \mathcal{F}(\mathcal{C} \times \underline{\mathcal{G}}^2, \Omega^0(P)),\\
&\tilde{\psi}^a \doteq \tilde{\eta}_1^{a,1} \in \mathcal{H}^{(1,1)} \subset \mathcal{F}(\mathcal{C} \times \underline{\mathcal{G}}^1, \Omega^1(P)), && \tilde{B}^a \doteq \tilde{\eta}_2^{a,0} \in \mathcal{H}^{(2,0)} \subset \mathcal{F}(\mathcal{C}, \Omega^2(P)),\\
&\tilde{\eta}_i^{a,2-i} \in \mathcal{H}^{(i,2-i)} \subset \mathcal{F}(\mathcal{C} \times \underline{\mathcal{G}}^{*i-2}, \Omega^i(P))
\end{aligned}$$

and they generate a bigraded differential algebra $\mathcal{H} = \oplus_{(m,n) \in Z^+ \times Z} \mathcal{H}^{(m,n)}$. The algebraic connection and its curvature are elements:

$$\begin{aligned}
&\tilde{\omega} \in \mathcal{G} \otimes \mathcal{H}^1 \doteq \mathcal{G} \otimes (\mathcal{H}^{(0,1)} \oplus \mathcal{H}^{(1,0)} \oplus \cdots \oplus \mathcal{H}^{(q,1-q)}),\\
&\tilde{\varrho} \in \mathcal{G} \otimes \mathcal{H}^2 \doteq \mathcal{G} \otimes (\mathcal{H}^{(0,2)} \oplus \mathcal{H}^{(1,1)} \oplus \mathcal{H}^{(2,0)} \oplus \cdots \oplus \mathcal{H}^{(q,2-q)}).
\end{aligned}$$

The δ operator is a $(1, -1)$-bigraded derivation on $\mathcal{H}$ and e^δ defines homomorphisms:

$$\begin{aligned}
&\mathrm{e}^\delta : \mathcal{G} \otimes \mathcal{H}^{(0,1)} \to \mathcal{G} \otimes \mathcal{H}^1, \qquad \mathrm{e}^\delta : \mathcal{G} \otimes \mathcal{H}^{(0,2)} \to \mathcal{G} \otimes \mathcal{H}^2,\\
&\tilde{c} \to \mathrm{e}^\delta \tilde{c} = \tilde{\omega}, \quad \tilde{\phi} \to \mathrm{e}^\delta \tilde{\phi} = \tilde{\varrho},
\end{aligned}$$

which transforms

$$b\tilde{c} + \tfrac{1}{2}[\tilde{c}, \tilde{c}] = \tilde{\phi} \xrightarrow{\mathrm{e}^\delta} \tilde{d}\tilde{\omega} + \tfrac{1}{2}[\tilde{\omega}, \tilde{\omega}] = \tilde{\varrho}, \qquad b\tilde{\phi} + [\tilde{c}, \tilde{\phi}] = 0 \xrightarrow{\mathrm{e}^\delta} \tilde{d}\tilde{\varrho} + [\tilde{\omega}, \tilde{\varrho}] = 0.$$

8. Concluding remarks

(1) Our model extends the original TYMT defined for positive ghost number fields to more general models containing negative ghost number fields as well. The main ideas behind one and another formulation is to accommodate the fields either as components of a connection with total degree 1 or as components of a curvature which has total degree 2. Nonetheless, in the process of obtaining Witten's action for TYMT as the gauge fixing of the symmetries of the classical action $\int \mathrm{Tr}\, F \wedge F$ [2,24] we have to introduce other fields with total degree other than 1 or 2 that cannot be components of $\mathcal{W}$ or $\mathcal{F}$. We can, however, define other ladders in order to accommodate those fields in the same way as

it was done in [6]. For example, for fields with total degrees -1 and 0 it is possible to introduce two ladders $\mathcal{B} = \sum_i \theta_i^{-1-i}$, $\Psi = \sum_i \lambda_i^{-i}$ and impose BRST transformations from $\tilde{d}\mathcal{B} + [\mathcal{W}, \mathcal{B}] = \Psi$. Then, we can develop our model following the same procedure of Section 2. Other choices of ladders and transformations are possible and will depend on what type of model one intends to build.

(2) A parallel development that is close to ours, and that presents an equivalent form of equations (26) and (27), was proposed in [13] in the study of two- and four-dimensional topological matter. In fact, the operators δ and b satisfying $[\delta, b] = d$ and $[b, d] = 0$ suggest that they are related to the odd generators G_μ and Q of the topological algebra. Here, identifying $\delta = \delta_\mu \otimes dx^\mu \leftrightarrow G = G_\mu \otimes dx^\mu$ and $-b \leftrightarrow Q$ we obtain that $[G, Q] = d$, $[Q, d] = 0$. In addition to these relations, we may have models with either $[\delta, d] = 0$ or $[\delta, d] \neq 0$ which would correspond to $[G, d] = 0$ or $[G, d] \neq 0$. This last possibility, however, does not appear in the topological algebras of [13]. Since $[\delta, d] = \Delta_2^{-1}$, it may be possible to have topological algebras with extra generators $\Delta_k^{1-k} = 1/k![\delta, \Delta_{k-1}^{2-k}]$, $k = 2, \dots, D$. The existence in [13] of a set of descendents fields given by $\phi^{(n)}_{\mu_1\mu_2\cdots\mu_n}(x) = 1/n![G_{\mu_1}, [G_{\mu_2} \dots [G_{\mu_n}, \phi(x)] \dots]]$ is equivalent to the imposition of (26) and (27). A quite similar approach was presented in [14] in the study of balanced topological field theory. Despite these analogies, the details behind one and another formulation are completely different. In [28] we show how to construct topological algebras for models defined by ladders (1) and (2) and derivative $\tilde{d} = b + d$. In particular, by taking the case of two dimensions we also show how the δ operator induces a supersymmetry algebra.

(3) It may be possible to interpret our model in terms of equivariant cohomology. First, we introduce the Weil algebra $W(\mathcal{G}) \doteq S(\mathcal{G}^*) \otimes \bigwedge(\mathcal{G}^*)$ where we assume c^a as the odd generators of degree 1, and ϕ^a as the even generators of degree 2. The differential in $W(\mathcal{G})$ is defined as $d_W c^a = -f^a_{bc} c^b c^c + \phi^a$, $d_W \phi^a = -f^a_{bc} c^b \phi^c$. In the construction of [29,30], TYMT is understood in terms of the BRST model for equivariant cohomology, i.e. as a differential algebra (B, d_B) with $B = (W(\mathcal{G}) \otimes \Omega(M))_{\text{basic}}$ the subalgebra of $W(\mathcal{G}) \otimes \Omega(M)$ invariant by the action of $I_a \otimes 1 + 1 \otimes I_a$ and $L_a \otimes 1 + 1 \otimes L_a$ (we denote by $I_a \otimes 1$ and $L_a \otimes 1$ the action of the interior derivative and the Lie derivative on $W(\mathcal{G})$, and $1 \otimes I_a$ and $1 \otimes L_a$ the respective action on $\Omega(M)$). The differential is $d_B = d_W \otimes 1 + 1 \otimes d_M + c^a \otimes L_a - \phi^a \otimes I_a$. Since the generators of B contain only the positive ghost number fields c^a and ϕ^a there is no possibility to introduce negative ghost number fields in B. A solution would be to replace $\Omega(M)$ by an appropriate $\mathcal{G}$-algebra $\mathcal{B}$ such that $W(\mathcal{G}) \otimes \mathcal{B}$ would accommodate the negative ghost number fields. In this approach, the BRST operator is considered as the differential in the algebra $B = W(\mathcal{G}) \otimes \mathcal{B}$ [29,31]. The problem then reduces to find an appropriate differential for B so that it gives the correct transformations for all the fields. The increasing complexity of the transformations of negative ghost number fields make this program difficult to be implemented.

(4) We have seen that $b\,\mathrm{Tr}\,\phi^N = 0 \overset{e^\delta}{\Rightarrow} \tilde{d}\,\mathrm{Tr}\,\mathcal{F}^N = 0 \rightleftharpoons (b + d)\mathrm{Tr}\,\mathcal{F}^N + \Delta\,\mathrm{Tr}\,\mathcal{F}^N = 0$. $\mathrm{Tr}\,\mathcal{F}^N$ is the Nth Chern class with $\mathcal{F}$ given by (2). In the problem of cohomology of b (modulo d) $(b + d)\hat{\Omega}^{(2N)} = 0$, the solution $\hat{\Omega}^{(2N)}$ does not coincide with $\mathrm{Tr}\,\mathcal{F}^N$ (unless $\Delta = 0$). This is a major difference from the results of [1–3] where the Chern class

Tr $\mathcal{F}^N$ (being a solution of descent equations) also belonged to the cohomology of b modulo d. In our model, when $[\delta, d] \neq 0$, $\hat{\Omega}^{(2N)}$ and Tr $\mathcal{F}^N$ will not agree. A direct consequence of this was observed in the model of Section 3, as it is explicitly seen in the differences between (110)–(114) and (7).

Acknowledgements

The author thanks Professors Tosiaki Kori and Kimio Ueno of Waseda University for many helpful discussions on this work, and the support from Professors Jose A. Helayel (CBPF), Sebastião A. Dias (CBPF) and Alexandre L. Oliveira (GFAR-UFRJ). The author also thanks Japanese monbusho for the scholarship, Miss Miho Tanaka from International center of Waseda university, Aurelina Carvalho,Ying Chen, Teodora Pereira, Jose E. Carvalho, IC XC Nika.

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